COMPACTNESS IN POINTFREE TOPOLOGY

by

NDUDUZO TEDIUS TWALA

DISSERTATION
Submitted in fulfilment of the
requirements for the degree of

MASTER OF SCIENCE
in
MATHEMATICS

in the

FACULTY OF SCIENCE AND AGRICULTURE
(School of Mathematical and Computer Sciences)

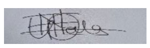
at the

UNIVERSITY OF LIMPOPO
SUPERVISOR: DR M.Z. MATLABYANA

2022

Declaration

I declare that the dissertation submitted to the University of Limpopo, for the degree of Master of Science has not previously been submitted by me for a degree at this or any other university; that it is my own work and that all the sources that I have used or quoted have been indicated and acknowledged by means of complete references.

Signature:  Date: 21 December 2021

Acknowledgement

Firstly, I would like to thank God for keeping me and giving me strength to get to this point of my life. Secondly, I would like to thank my supervisor Dr. Mack Matlabyana for his patience and support towards me. I thank him not only for helping me complete my research, but also for passionately showing and guiding me on how research is conducted in this field of Mathematics. I also extend my sincere gratitude to the following colleagues:Ms Annah Takalani and Mr Thabo Malatji; I am grateful for the selfless decision they tookto give me some time off so as to focus on finishing this research.

I thank the University of Limpopo for giving me unlimited access to the internet and Library to explore more information on my field of study. Lastly, I thank my family for the support they gave me through out this journey of my academic adventure: especially my aunt Arsian Twala for her unconditional love, words of encouragement and for always keeping me in her prayers.

Abstract

Our discussion starts with the study of convergence and clustering of filters initiated in pointfree setting by Hong, and then characterize compact and almost compact frames in terms of these filters. We consider the strict extension and show that $t_Q L$ is a zero-dimensional compact frame, where Q denotes the set of filters in L. Furthermore, we study the notion of general filters introduced by Banaschewski and characterize compact frames and almost compact frames using them. For filter selections, we consider $\mathrm{F}-$compact and strongly $\mathrm{F}-$compact frames and show that lax retracts of strongly $\mathrm{F}-$compact frames are also strongly $\mathrm{F}-$compact. We study further the ideals $\mathsf{R}_s(L)$ and $\mathsf{R}_K(L)$ of the ring of real-valued continuous functions on L, $\mathsf{R}L$. We show that $\mathsf{R}_s(L)$ and $\mathsf{R}_K(L)$ are improper ideals of $\mathsf{R}L$ if and only if L is compact. We consider also fixed ideals of $\mathsf{R}L$ and showthat L is compact if and only if every ideal of $\mathsf{R}L$ is fixed if and only if every maximalideal of $\mathsf{R}L$ is fixed. Of interest, we consider the class of isocompact locales, which is larger that the class of compact frames. We show that isocompactness is preserved by nearly perfect localic surjections. We study perfect compactifications and show that the Stone-Čech compactifications and Freudenthal compactifications of rim-compact frames are perfect. We close the discussion with a small section on $Z-$closed frames and show that a basically disconnected compact frame is $Z-$closed.

Contents

Chapter 1

Introduction and preliminaries

1.1 Synopsis of the dissertation

The notion of compactness is one of the most studied concepts both in classical and pointfree topology. The formal definition of compactness in spaces was brought by Fréchet (see [28]), and in frames compactness was studied by several authors namely Banaschewski, Mulvey, Hong, and Baboolal just to name a few. The purpose of this dissertation is to collate and study variants of compactness in pointfree setting.

The dissertation consist of 7 chapters, and is structured as follows: In the first chapter, we give some preliminary definitions, notations and basic results to be used in the upcoming chapters. Chapter 2 introduces the concept of compactness in frames and is characterised in terms of convergence and clustering of filters. This idea was introduced by Hong in [30]. We also discuss the concept of almost compactness and show that in a regular frame, these two concepts coincide.

In chapter 3, we first discuss strict extensions of frames and give some of the properties of strict extensions. In the second section of this chapter we use the paper by Banaschewski and Hong [9] to construct strict extensions of frames related to a set of filters. This construction was first introduced by Hong in [30] where he also constructed a zero-dimensional compactification of a zero-dimensional frame. We end this chapter with a constructive compactification of completely regular frames.

The work in chapter 4 is based on the papers [16] and [12]. The focus is mainly about general filters on frames, which were introduced by Banaschewski in [7]. Just as in [30], weuse general filters to characterise compactness and almost compactness. The third section in this chapter is based on the paper by Banaschewski and Hong in [12], where the notion of F-compactness is introduced.

In chapter 5 we study ideals in $\mathsf{R}L$, the ring of real-valued continuous functions on a regular frame L. The concept of weakly spatial frames is introduced and we show that fixed and strongly fixed ideals are equivalent in the case of weakly spatial frames.

The work in chapter 6 is based on the paper [23]. Dube, Naidoo and Ncube introduced the concept of isocompactness in the category of locales. In chapter 7, we first discuss perfect compactifications of frames. This work was introduced by Baboolal in [2].

1.2 Definitions and preliminary results

Definition 1.2.1. A *partially ordered* set is a set P with a binary relation $\leq$ such that for all $x, y, z \in P$, the following conditions are satisfied:

i. Reflexivity: $x \leq x$;

ii. Antisymmetry: if $x \leq y$ and $y \leq x$; then $x = y$;

iii. Transitivity: if $x \leq y$ and $y \leq z$; then $x \leq z$.

Definition 1.2.2. Let A and B be partially ordered sets. Monotone increasing maps $f : A \to B$ and $g : B \to A$ are said to be *(Galois) adjoint* if $f(y) \leq x$ if and only if $g(x) \leq y$. This condition is equivalent to $fg(y) \leq y$ and $gf(x) \geq x$ for all $x \in A$, $y \in B$. We say f is the *left adjoint* of g and g is the *right adjoint* of f.

The *lower bound* of a nonempty subset S of a partially ordered set P is an element $l \in P$ such that $l \leq s$ for all $s \in S$. Similarly, the *upper bound* of a subset S of a partially ordered set P is an element $u \in P$ such that $u \geq s$ for all $s \in S$. A partially ordered set P is called a *lattice* if every pair of elements has a unique least upper bound (supremum) and a unique greatest lower bound (infimum). A lattice L is called a *complete lattice* if every subset S of L has a supremum and infimum in L.

Definition 1.2.3. A *frame* is a complete lattice L satisfying the distributive law

$$a \wedge (\bigvee S) = \bigvee \{a \wedge s \mid s \in S\}, \text{ where } a \in L \text{ and } S \subseteq L.$$

We denote the bottom element of a frame L by 0 (or 0_L), and the top element by 1 (or 1_L).

One example of a frame comes from the lattice of open subsets of a topological space X, denoted $\mathsf{O}(X)$ and ordered by set inclusion $\subseteq$ with finite meets and arbitrary joins given by intersection and union of sets, respectively.

Definition 1.2.4. Let L and M be frames. Then a map $h : L \to M$ is called a *frame homomorphism* if h preserves finite meets (including 1) and arbitrary joins (including 0). That is:

i. $h(a \wedge b) = h(a) \wedge h(b)$ for all $a, b \in L$;

ii. $h(\bigvee S) = \bigvee h(S)$ for any $S \subseteq L$;

iii. $h(0) = 0$ and $h(1) = 1$.

For any topological spaces X and Y, a continuous map $f : X \to Y$ induces a frame homomorphism of $\mathsf{O}(f) \colon \mathsf{O}(Y) \to \mathsf{O}(X)$ which maps $U \in \mathsf{O}(Y)$ to $f^{-1}(U) \in \mathsf{O}(X)$.

For any frame homomorphism $h \colon L \to M$, we say h is *dense* if $h(a) = 0$ implies $a = 0$ and we say h is *codense* if $h(a) = 1$ implies $a = 1$.

Definition 1.2.5. For any frame homomorphism $h \colon L \to M$ there exists a map $h_* \colon M \to L$, called the *right adjoint* of h, defined by $h_*(b) = \bigvee \{a \in L \colon h(a) \leq b\}$.

The right adjoint of a frame homomorphism is not necessarily a frame homomorphism, but it preserves arbitrary meets. We observe that h is onto if and only if $hh_* = id_M$.

Definition 1.2.6. An element p in a frame L is said to be *prime* if $p < 1$ and $a \wedge b \leq p$ implies $a \leq p$ or $b \leq p$.

Denote by ΣL, the set of all prime elements of L. The set ΣL is called the *spectrum* of L.

Definition 1.2.7. A frame L is *spatial* if for all $a, b \in L$ such that $a \nleq b$, there exists $p \in \Sigma L$ such that $a \leq p$ but $b \nleq p$.

Definition 1.2.8. Let L be a frame. A *nucleus* is a map $j : L \to L$ such that for all $a, b \in L$:

1. $a \leq j(a)$;
2. $j(a \wedge b) = j(a) \wedge j(b)$;
3. $j^2(a) = j(a)$.

The set $Fix(j) = \{x \in L \mid j(x) = x\}$ is a frame with meet as in L and join $j(\bigvee S)$ for each $S \subseteq Fix(j)$. Furthermore, $j : L \to Fix(j)$ is a quotient map with the inclusion $Fix(j) \hookrightarrow L$ as its right adjoint.

Definition 1.2.9. The *pseudocomplement* of an element a in a frame L is the element given by $a^* = \bigvee\{ x \in L \mid x \wedge a = 0\}$.

If $a^* = 0$ in a frame L , we say a is a *dense* element, and we shall denote by $D(L)$ the set of all dense elements in the frame L. Furthermore; for a frame L, the set $\mathcal{B}L = \{a \in L \mid a^{**} = a\}$ is called the *Booleanization* of L. In fact, $\mathcal{B}L$ forms a frame.

Definition 1.2.10. We call an element a in a frame L *complemented* if there exists an element $a' \in L$ such that $a \wedge a' = 0$ and $a \vee a' = 1$.

Throughout this work, we shall denote by $C(L)$ the set of all complemented elements of a frame L.

Well below, rather below and completely below relations

Definition 1.2.11. In a frame L, we say an element a is *well below* $b \in L$, written $a \ll b$, if for any $S \subseteq L$ with $b \leq \bigvee S$, there exists a finite $T \subseteq S$ such that $a \leq \bigvee T$.

Definition 1.2.12. In a frame L, we say an element $a \in L$ is *rather below* an element $b \in L$, written $a \prec b$, if $a^* \vee b = 1$ or equivalently,

$a \prec b$ if and only if there exists $c \in L$ such that $a \wedge c = 0$ and $c \vee b = 1$ (c is called the *separating element*).

Lemma 1.2.1. *In a frame L, the following properties are satisfied for $\prec$:*

1. $a \prec b$ *implies* $a \leq b$.
2. $0 \prec a \prec 1$ *for all* $a \in L$.
3. $x \leq a \prec b \leq y$ *implies* $x \prec y$.
4. *If* $a \prec b$ *then* $b^* \prec a^*$.
5. *If* $a \prec b$ *then* $a^{**} \prec b$.
6. *If* $a_1, a_2 \prec b_1, b_2$, *then* $a_1 \vee a_2 \prec b_1 \vee b_2$ *and* $a_1 \wedge a_2 \prec b_1 \wedge b_2$.

Definition 1.2.13. Let L be a frame. Then we say an element a is *completely below* an element b, written $a \prec\prec b$, if there are $a_r \in L$ (r rational, $0 \leq r \leq 1$) such that $a_0 = a$, $a_1 = b$ and $a_r \prec a_s$ for $r < s$.

Lemma 1.2.2. *In a frame L, the following properties are satisfied for $\prec\prec$:*

1. $a \prec\prec b$ *implies* $a \prec b$.
2. $a \prec\prec b$ *implies* $a \leq b$.
3. $0 \prec\prec a \prec\prec 1$ *for all* $a \in L$.
4. $x \leq a \prec\prec b \leq y$ *implies* $x \prec\prec y$.
5. *If* $a \prec\prec b$ *then* $b^* \prec\prec a^*$.

6. *If* $a \prec\prec b$ *then* $a^{**} \prec\prec b$.

7. *If* $a_1, a_2 \prec\prec b_1, b_2$, *then* $a_1 \vee a_2 \prec\prec b_1 \vee b_2$ *and* $a_1 \wedge a_2 \prec\prec b_1 \wedge b_2$.

Proposition 1.2.1. *The relation* $\prec\prec$ *is interpolative, that is, for all* $a \in L$: $a \prec\prec b$ *implies there exists* $c \in L$ *such that* $a \prec\prec c \prec\prec b$.

$\prec\prec$ is the largest interpolative relation contained in $\prec$.

Definition 1.2.14. We say a frame L is:

1. *continuous* if $a = \bigvee\{x \in L \mid x \ll a\}$ for all $a \in L$.
2. *regular* if $a = \bigvee\{x \in L \mid x \prec a\}$ for all $a \in L$.
3. *completely regular* if $a = \bigvee\{x \in L \mid x \prec\prec a\}$ for all $a \in L$.

Ideals and filters

For an element a in a frame L, set $\uparrow a = \{x \in L \mid x \geq a\}$ and $\downarrow a = \{x \in L \mid x \leq a\}$. Then we say a subset $U \subseteq L$ is an *up-set* if $\uparrow U = \bigcup\{\uparrow a \mid a \in U\} = U$ and we say $D \subseteq L$ is a *down-set* if $\downarrow D = \bigcup\{\downarrow a \mid a \in D\} = D$. We shall denote by $\mathcal{D}L$ the set of all down-sets ofa frame L.

Definition 1.2.15. An *ideal* in a frame L is a subset $I \subseteq L$ such that :

i. $0 \in I$;

ii. $a, b \in I$ implies $a \vee b \in I$;

iii. $b \leq a$ and $a \in I$ implies $b \in I$.

Proposition 1.2.2. *The set* $I(L)$ *of all ideals in a frame* L*, ordered by the inclusion, is a frame.*

Proof. Let $(I_j)_{j\in J}$ be a collection of ideals in L.

i. $0 \in I_j$ for all $j \in J$ implies $0 \in \bigcap_{j\in J} I_j$;

ii. $a, b \in \bigcap_{j\in J} I_j$ implies $a, b \in I_j$ for all $j \in J$ which implies $a \vee b \in I_j$ for all $j \in J$ and thus $a \vee b \in \bigcap_{j\in J} I_j$;

iii. $a \leq b$ and $b \in \bigcap_{j\in J} I_j$ implies $b \in I_j$ for all $j \in J$ which implies $a \in I_j$ for all $j \in J$ and thus $a \in \bigcap_{j\in J} I_j$.

Hence the intersection of ideals is an ideal. Now for the join of a collection of ideals $(I_j)_{j\in J}$, define

$\bigvee_{j\in J} I_j = \{\bigvee X \mid X \text{ finite}, X \subseteq \bigcup_{j\in J} I_j\}$. We show that this set is an ideal.

Take $\bigvee X, \bigvee Y \in \bigvee_{j\in J} I_j$, then
$(\bigvee X) \vee (\bigvee Y) = \bigvee\{x \vee y \mid x \in X, y \in Y\} \in \bigvee_{j\in J} I_j$. Since $\{x \vee y \mid x \in X, y \in Y\}$ is a finite subset of $\bigcup_{j\in J} I_j$. Now, let $a \leq \bigvee X$ for some $a \in L$ and $\bigvee X \in \bigvee_{j\in J} I_j$. Then
$a = a \wedge (\bigvee X) = \bigvee\{a \wedge x \mid x \in X\}$, and $\{a \wedge x \mid x \in X\}$ is a finite subset of $\bigcup_{j\in J} I_j$, hence $a \in \bigvee_{j\in J} I_j$.

If K is an ideal containing all of the I_j, then $\bigvee X \in K$ for any finite $X \subseteq \bigcup_{j\in J} I_j$.

Now let $(I_j), K$ be ideals in L. If $a \in (\bigvee_{j\in J} I_j) \cap K$, then $a = \bigvee X$ for some finite

$X \subseteq \bigcup_{j\in J} I_j$. But $X \subseteq \bigcup_{j\in J}(I_j \cap K)$, hence $a = \bigvee X \in \bigvee_{j\in J}(I_j \cap K)$. Conversely,

$$\begin{aligned}
\bigvee_{j\in J}(I_j \cap K) &= \{\bigvee X \mid X \text{ finite}, X \subseteq \bigcup_{j\in J}(I_j \cap K)\} \\
&= \{\bigvee X \mid X \text{ finite}, X \subseteq (\bigcup_{j\in J} I_j) \cap K\} \\
&\subseteq \{\bigvee X \mid X \text{ finite}, X \subseteq \bigcup_{j\in J} I_j\} \cap K \\
&= (\bigvee_{j\in J} I_j) \cap K.
\end{aligned}$$

Hence $I(L)$ is a frame. ∎

Definition 1.2.16. A *filter* in a frame L is a subset $F \subseteq L$ such that :

i. $1 \in F$ and $0 \notin F$;

ii. $a, b \in F$ implies $a \wedge b \in F$;

iii. $b \geq a$ and $a \in F$ implies $b \in F$.

Definition 1.2.17. We say a filter F in a frame L is:

1. *prime* if $x \vee y \in F$ implies $x \in F$ or $y \in F$ for each $x, y \in L$;
2. *completely prime* if $\bigvee S \in F$ implies $S \cap F \neq \emptyset$ for each $S \subseteq L$;
3. *Boolean* if for each $x \in F$, there is $y \in F \cap C(L)$ such that $y \leq x$;
4. an *ultrafilter* if whenever G is a filter in L with $F \subseteq G$, then $F = G$.

Lemma 1.2.3. *If F is a filter in a frame L, then the set*

$$secF = \{x \in L \mid x \wedge a \neq 0 \textit{ for all } a \in F\} \textit{ is also a filter.}$$

Proof. Since $0 \wedge a = 0$ for all $a \in F$, then $0 \notin secF$ and $1 \in secF$ since $1 \wedge a \neq 0$ for all $a \in F$. Now let $x, y \in secF$, then $x \wedge a \neq 0$ and $y \wedge a \neq 0$ for all $a \in F$. Therefore

$$\begin{aligned}(x \wedge y) \wedge a &= (x \wedge a) \wedge (y \wedge a) \\ &\neq 0 \text{ for all } a \in F \text{ implies } x \wedge y \in secF.\end{aligned}$$

Lastly, if $x \in secF$ and $x \leq y$ for some $y \in L$, then $x \wedge a \leq y \wedge a$ for all $a \in F$. But $x \wedge a \neq 0$ which implies that $y \wedge a \neq 0$ and hence $y \in secF$. Thus, $secF$ is a filter. $\square$

Furthermore, we observe that $F \subseteq secF$ since $x \wedge y \neq 0$ for all $x, y \in F$.

Chapter 2

Compactness in frames

In this chapter we study convergence and clustering of filters and characterise compact frames and almost compact frames in terms of these filters. Compactness plays an important role both in classical topology and pointfree topology. Our interest is to focus on compactness in the area of pointfree topology.

2.1 Background

While Fréchet [28] introduced the definition of compactness in spaces, Alexandroff and Urysohn [1] generalized this definition in terms of open covers in spaces. This generalization then gave rise to the study of compactness in frame setting, and several authors have since studied and expanded this concept (see [7], [30], [12], [20]).

Definition 2.1.1. A cover of a frame L is a subset $A \subseteq L$ such that $\bigvee A = 1$.

Definition 2.1.2. A frame L is said to be *compact* if every cover of L has a finite subcover. That is, L is compact if for every subset $A \subseteq L$ with $\bigvee A = 1$, there exists a finite subset $B \subseteq A$ with $\bigvee B = 1$.

The following example is taken from [35] on page 131.

Example 2.1.1. [35] The set $I(L)$ of all ideals in a frame L is compact. Let $(I_j)_{j\in J}$ be a cover of $I(L)$. Then $1 \in L = 1_{I(L)} = \bigvee_{j\in J} I_j$. Hence there exists a finite $X \subseteq \bigcup_{j\in J} I_j$ such that $1 = \bigvee X$. But since X is finite, $X \subseteq \bigcup_{j\in J_0} I_j$ for some finite $J_0 \subseteq J$. Hence

$$1 = \bigvee X \in \bigvee_{j\in J_0} I_j \implies 1_{I(L)} = L = \bigvee_{j\in J_0} I_j.$$

Proposition 2.1.1. *If L is a compact frame, then the up-set $\uparrow a$ of any element a in L is compact.*

Proof. Suppose L is compact and let $a \in L$. Take any cover A of $\uparrow a$, then $\bigvee A = 1_{\uparrow a} = 1_L$. This means that A is a cover of L, and since L is compact there exists a finite $B \subset A$ such that $\bigvee B = 1_L = 1_{\uparrow a}$. Hence $\uparrow a$ is compact. $\square$

Observation 2.1.1. *If $a \leq b$ and $\uparrow a$ is compact in a frame L, then $\uparrow b$ is compact. To see this, we note that $\uparrow a \supseteq \uparrow b$ and a cover for $\uparrow a$ is also a cover for $\uparrow b$.*

Lemma 2.1.1. *[19] Let a and b be elements of a frame L. If $\uparrow a$ and $\uparrow b$ are both compact, then $\uparrow(a \wedge b)$ is compact.*

Proof. Let C be a cover of $\uparrow(a\wedge b)$. Then $\{a\vee x \mid x \in C\}$ is a cover of $\uparrow a$ and $\{b\vee x \mid x \in C\}$ is a cover of $\uparrow b$. By compactness of $\uparrow a$ and $\uparrow b$, there exists finite $S, T \subseteq C$ such that $a \vee \bigvee S = 1$ and $b \vee \bigvee T = 1$. Therefore

$$(a \vee \bigvee S) \wedge (b \vee \bigvee T) = 1.$$

Applying the distributivity law repeatedly we arrive at

$$(a \wedge b) \vee (b \wedge \bigvee S) \vee (a \wedge \bigvee T) \vee (\bigvee S \wedge \bigvee T) = 1.$$

Each of the terms in the brackets is below $\bigvee(S \cup T)$, consequently $\bigvee(S \cup T) = 1$ $\square$

If the codomain of a codense frame homomorphism $h: L \to M$ is compact, then we know that the source is also a compact frame. We capture this nice result in the following theorem.

Theorem 2.1.1. *Let $h : L \to M$ be a codense frame homomorphism. If M is compact, then L is also compact.*

Proof. Let A be a cover of L, then $\bigvee A = 1_L$ and since frame homomorphisms preserve top elements, it follows that $\bigvee h(A) = h(\bigvee A) = h(1_L) = 1_M$. This means that $h(A)$ is a cover of M. M is compact, therefore there exists a finite $B \subseteq h(A)$ such that $\bigvee B = 1_M$; say $B = \{h(a_1), ..., h(a_n)\}$. Now, $1_M = \bigvee\{h(a_1), ..., h(a_n)\} = \bigvee_{i=}^{n} h(a_i) = h(\bigvee_{i=}^{n} a_i)$ which implies $\bigvee_{i=}^{n} a_i = 1_L$ since h is codense. Thus $\{a_1, ..., a_n\}$ is a finite subcover of A, and hence L is compact.

$\square$

2.2 Convergence of filters in frames

In this section, we define the convergence and clustering of a filter introduced in [30] by S.S Hong.

Definition 2.2.1. A filter F in a frame L is said to be:

1. *convergent* if F meets every cover C of L.
2. *clustered* if $secF$ meets every cover C of L.

Lemma 2.2.1. *[30] Let F be a filter in a frame L.*

1. *If F is completely prime, then F is convergent;*

2. *If F is convergent, then F is clustered;*

3. *A filter containing a completely prime filter is convergent;*

Proof. 1. Suppose F is a completely prime filter in L, then for any cover C of L, $\bigvee C \in F$ implies $C \cap F \neq \emptyset$ since F is completely prime and hence F is convergent.

2. Suppose F is convergent filter in L, then for any cover C of L, $C \cap F \neq \emptyset$. Hence $C \cap secF \neq \emptyset$ since $F \subseteq secF$. Therefore F is clustered.

3. Suppose F is a filter in L containing a completely prime filter G, then for any cover C of L, $\bigvee C \in G$ implies $C \cap G \neq \emptyset$ and hence $C \cap F \neq \emptyset$.

∎

Lemma 2.2.2. *[30] If F is a maximal filter in a frame L, then F is convergent if and only if it is clustered.*

Proof. (⇒): Let F be a convergent maximal filter in a frame L and C be any cover of L. Then $C \cap F \neq \emptyset$, since F is convergent. But F is maximal and therefore $F = secF$ implies $secF \cap C \neq \emptyset$, and hence F is clustered.
(⇐): Suppose F is a clustered maximal filter in L. Then for any cover C of L, $secF \cap C \neq \emptyset$ implies $C \cap F \neq \emptyset$ since $F = secF$. Hence, F is convergent. ∎

Recall that in a frame L, we say $A \subseteq L$ *refines* $B \subseteq L$ if for any $a \in A$, there exists $b \in B$ such that $a \leq b$. It is also worth noting that If C is a cover of L that refines B, then B is also a cover of L.

Proposition 2.2.1. *[30] A filter F in a frame L is clustered if and only if*

$$\bigvee\{x^* \mid x \in F\} = 1.$$

Proof. (⇒): Suppose F is a clustered filter in the frame L, then $secF$ meets every cover of L. If $\bigvee\{x^* \mid x \in F\} = 1$, then there exists $x' \in L$ such that $x' \in secF \cap \{x^* \mid x \in F\}$ which implies that $x' \in secF$ and $x' \in \{x^* \mid x \in F\}$. But $x' \in secF$ implies $x' = x^*$ such that $x \in F$ and this implies that $x' = x^* \in secF$ and $x \in F$ implies $x' \wedge x = x^* \wedge x \neq 0$ which is a contradiction.

(⇐): Suppose $\bigvee\{x^* \mid x \in F\} \neq 1$. If C is a cover of L such that $secF \cap C = \emptyset$, then for any $c \in C$ there exists $x \in F$ such that $c \wedge x = 0$. Which implies $c \leq x^*$ meaning C refines $\{x^* \mid x \in F\}$ and since C is a cover of L, this means that $\{x^* \mid x \in F\}$ is also a cover of L, contrary to the assumption $\bigvee\{x^* \mid x \in F\} \neq 1$. Thus $secF$ meets every cover of L, and therefore F is clustered. $\square$

Theorem 2.2.1. *[20] A frame is compact if and only if each of its prime up-sets converges.*

Proof. (⇒): Suppose L is a compact frame, $U \subseteq L$ a prime upset and $C \subseteq L$ a cover of L. Then there exists a finite $K \subseteq C$ such that $\bigvee K = 1$. But $1 \in U$ implies $\bigvee K \in U$ which implies $K \cap U \neq \emptyset$, and hence $C \cap U \neq \emptyset$.

(⇐): Suppose that each prime upset in the frame L converges and let C be a cover of L. If C has no finite subcover, then

$$U = \{x^* \in L \mid x \nleq \bigvee S \text{ for any finite } S \subseteq C\}$$

is an upset and $U \cap C = \emptyset$ which is a contradiction. $\square$

Next we consider the class of almost compact frames, we start with the following definition.

Definition 2.2.2. A frame L is *almost compact* if for any cover C of L, there is a finite $T \subseteq C$ with $(\bigvee T)^* = 0$.

In the definition above, we observe that $(\bigvee T)^* = 0$ which implies that $(\bigvee T)^{**} = 1$, that is, $\bigvee T$ is dense.

Remark 2.2.1. Compactness implies almost compact since for any cover C of L with a finite subcover T, $\bigvee T = 1$. Then $(\bigvee T)^* = 0$ and hence almost compact. Therefore the class of compact frames is contained in the class of almost compact frames.

Proposition 2.2.2. *[30] A frame L is almost compact if and only if for any filter F in L, $\bigvee\{x^* \mid x \in F\} = 1$.*

Proof. ($\Rightarrow$): Suppose L is almost compact and let F be any filter in L. If $\bigvee_{i\in I}\{x_i^* \mid x_i \in F\} = 1$, then there exists $\{x_i^* \mid x_i \in F,\ i = 1, 2, \ldots, n\} \subseteq \{x_i^* \mid x_i \in F, i \in I\}$ such that $(\bigvee_{i=1}^{n} x_i^*)^* = 0$ which implies that $\bigwedge_{i=1}^{n} x_i^{**} = 0$ implying $0 \in F$, which is a contradiction.
($\Leftarrow$): Suppose that for any filter F in L, $\bigvee\{x^* \mid x \in F\} = 1$. Then by Proposition 2.2.1 F is clustered. Thus for any cover S of L, we have $secF \cap S \neq \emptyset$. Pick a finite number of elements $x_{s_1}, x_{s_2}, ..., x_{s_n} \in secF \cap S$. Then for each $x_{s_i} \in secF \cap S$, there is $x_{s_i}^*$ such that $x_{s_i} \wedge x_{s_i}^* = 0$ and $x_{s_i} \vee x_{s_i}^*$ is dense. Now $T = \{x_{s_i} \vee x_{s_i}^* \mid i = 1, 2, ..., n\}$ is a finite set such that $(\bigvee T)^* = 0$. Thus L is almost compact. ∎

Corollary 2.2.1. *[30] For a frame L, the following are equivalent*:

1. *L is almost compact;*
2. *every filter in L is clustered;*
3. *every maximal filter in L is convergent.*

Proof. (1) $\Rightarrow$ (2): Suppose L is almost compact and F is a filter in L. If F does not cluster, then by Proposition 2.2.1, $\bigvee\{x^* \mid x \in F\} = 1$ which implies that $\{x^* \mid x \in F\}$ is a cover of L, and since L is almost compact, there exist $\{x_1^*, x_2^*, \ldots, x_n^*\} \subseteq \{x^* \mid x \in F\}$ such that $x_1^{**} \wedge x_2^{**} \wedge \ldots \wedge x_n^{**} = 0$. But $x \leq x^{**}$ for all $x \in F$ and therefore $x^{**} \in F$ which implies that $x_1^{**} \wedge x_2^{**} \wedge \ldots \wedge x_n^{**} \in F$, contrary to $0 \notin F$. Hence, F clusters.
(2) $\Rightarrow$ (3): This follows immediately from Lemma 2.2.2.

(3) ⇒ (1): If every maximal filter filter F in L converges. Then $F \cap C \neq \emptyset$ for any cover $C \subseteq L$. By Lemma 2.2.2 F is clustered and so $\bigvee\{x^* \mid x \in F\} \neq 1$. Then by Proposition 2.2.2 L is almost compact.

$\square$

Corollary 2.2.2. *[30] For a regular frame L, the following are equivalent:*

1. *L is compact;*
2. *every filter in L is clustered;*
3. *every maximal filter in L is convergent.*

Proof. It follows immediately from Proposition 2.2.2. $\square$

We show below that in a regular frame, compactness and almost compactness coincide.

Proposition 2.2.3. *[30] A regular almost compact frame is compact.*

Proof. Suppose a regular frame L is almost compact and let $\left(\bigvee\{x_i \in L \mid i \in I\}\right) = 1$. Since L is regular, $x_i = \bigvee\{y_{j_i} \prec x_i,\ j_i \in J_i\}$ and therefore $\bigvee\{y_{j_i} \prec x_i,\ j_i \in J_i,\ i \in I\} = 1$.$L$ is almost compact, therefore there exists a finite $K \subseteq \bigcup\{J_i \mid i \in I\}$ such that $\left(\bigvee\{y_k \mid k \in K\}\right)^{**} = 1$. Clearly, $\left(\bigvee\{y_k \mid k \in K\}\right)^{**} \prec \left(\bigvee\{x_i \mid k \in J_i \text{ for } k \in K\}\right)$, that is, $\bigvee\{x_i \mid k \in J_i \text{ for } k \in K\} = 1$ and hence L is compact. $\square$

Next we consider maps that transport filters, in particular we consider these maps which will map filters in the domain to filters in the codomain. These maps have been considered by Hong in [30].

Lemma 2.2.3. *[30]Let $h\colon L \to M$ be a frame homomorphism.*

1. *For any filter F in M, $h^{-1}(F)$ is again a filter in L.*

2. *If h is dense, then for any filter F in L, $h(F)$ is a filter base in M.*

3. *If h is dense, onto and F is a filter in L, then $h(F)$ is a filter in M.*

Proof. 1. For any frame homomorphism, $h(1) = 1 \in F$ so that $1 \in h^{-1}(F)$. Let $h(x)$, $h(y) \in F$. Since F is a filter, it follows that $h(x) \wedge h(y) \in F$ which impliesthat $h(x \wedge y) \in F$ so that $x \wedge y \in h^{-1}(F)$. That is, $x \wedge y \in h^{-1}(F)$ whenever $h(x) \wedge h(y) \in F$. Furthermore, if $x \in h^{-1}(F)$ and $x \leq y$ for some $y \in L$, then there exist $a \in F$ such that $x = h^{-1}(a)$ which implies that $a \leq h(y)$ and since F is a filter, then $h(y) \in F$. Hence $y \in h^{-1}(F)$, and $h^{-1}(F)$ is a filter.

2. Let $a = h(x)$, $b = h(y)$ for some $x, y \in F$ so that $a, b \in h(F)$. Since $x, y \in F$ and$F$ is a filter, it follows that $x \wedge y \in F$ and so that $x \wedge y \neq 0$. By denseness of h, $h(x \wedge y) = h(x) \wedge h(y) \neq 0$ and $h(x \wedge y) \leq h(x)$, $h(x \wedge y) \leq h(y) \in h(F)$. Therefore$h(F)$ is a filter base in M.

3. Since $1 \in F$, $h(1) = 1$ and therefore $1_M \in h(F)$ and $0_M = h(0) \notin h(F)$ since h is dense. Next, let $a, b \in h(F)$; then there exists $x, y \in F$ such that $a = h(x)$ and$b = h(y)$. Therefore $a \wedge b = h(x) \wedge h(y) = h(x \wedge b) \in h(F)$ since $x \wedge y \in F$. Lastly, let $a \leq b$ and $a \in h(F)$. Since h is onto, there exists $x, y \in L$ such that$a = h(x)$ and $b = h(y)$, that is, $h(x) \leq h(y)$. Again by ontoness of h, we have $h_* h(x) \leq h_* h(y)$ implying $x \leq y$ which implies $y \in F$ since $x \in F$. Thus $h(y) \in h(F)$. Therefore $h(F)$ is a filter in M whenever F is a filter in L.

□

The next two propositions show that convergent filters and clustered filters are preserved and reflected under the suitable mappings.

Proposition 2.2.4. *[30] Let $h \colon L \to M$ be a frame homomorphism.*

1. *If F is a convergent filter in M, then $h^{-1}(F)$ is also convergent in L.*

2. *Assume that h is dense, codense and onto, and a filter F in L is convergent,then $h(F)$ is also convergent.*

Proof. 1. Take any cover C of L, then $h(C)$ is a cover of M. Since F is convergent,we have $F \cap h(C) \neq \emptyset$. Take $x \in C$ with $h(x) \in F$, then $x \in h^{-1}(F) \cap C$. Hence $h^{-1}(F)$ is convergent.

2. Suppose C is a cover of M, then $h^{-1}(C)$ is again a cover of L since h is onto and codense. Therefore, there exists $x \in h^{-1}(C) \cap F$, which implies that $h(x) \in C \cap sec\ h(F)$, and hence $h(F)$ is convergent.

$\square$

Proposition 2.2.5. *[30] Let $h: L \to M$ be a frame homomorphism.*

1. *If F is a clustered filter in M, then $h^{-1}(F)$ is also clustered in L.*

2. *Assume that h is dense, codense and onto, and a filter F in L is clustered, then $h(F)$ is also clustered.*

Proof. 1. Take any cover C of L, then $h(C)$ is a cover of M. Since F is clustered, we have $secF \cap h(C) \neq \emptyset$. Take $x \in C$ with $h(x) \in secF$, then $x \in sec\ h^{-1}(F) \cap C$Hence $sec\ h^{-1}(F)$ is clustered.

2. Suppose C is a cover of M, then $h^{-1}(C)$ is again a cover of L since h is onto and codense. Therefore, there exists $x \in h^{-1}(C) \cap sec\ F$, which implies that $h(x) \in C \cap h(F)$, and hence $sec\ h(F)$ is clustered.

$\square$

Proposition 2.2.6. *[16] A filter F in a frame L is clustered if and only if for every cover C of L, there exists $x \in C$ such that $x^* \notin F$.*

Proof. $(\Rightarrow)$: Assume that F clusters and let C be any cover of L. If $x^* \in F$ $\forall x \in C$, then

$x^{**} \in \{y^* \mid y \in F\}$ and since $x \leq x^{**}$, we have

$$\begin{aligned} 1 &= \bigvee C \\ &= \bigvee \{x \mid x \in C\} \\ &\leq \bigvee \{x^{**} \mid x \in C\} \\ &\leq \bigvee \{a^* \mid a \in F\}, \text{ where } a = x^*, \end{aligned}$$

hence $\bigvee\{a^* \mid a \in F\} = 1$ and by Proposition 2.2.1, this contradicts the fact that F clusters.
$(\Leftarrow)$: Suppose that for any cover C of L and any filter F in L there exists $x \in C$ with $x^* \notin F$. Let F be a filter in L. If F does not cluster, then $\bigvee\{x^* \mid x \in F\} = 1$ which means $\{x^* \mid x \in F\}$ is a cover of L. Now, let $a \in \{x^* \mid x \in F\}$ such that $a^* \notin F$. But $a = x^*$ for some $x \in F$, which implies that $a^* = x^{**} \notin F$, a contradiction. $\square$

2.3 Compactness and separation in frames

We know that regularity does not imply normality and normality does not imply regularity in classical topology. This is indeed the case in pointfree setting. We show below that a compact frame is normal if the frame is regular.

Definition 2.3.1. A frame L is *normal* if for all $a, b \in L$ with $a \vee b = 1$, there exists $u, v \in L$ such that $u \wedge v = 0$ and $a \vee u = 1 = b \vee v$.

Theorem 2.3.1. *[35] A regular compact frame is normal.*

Proof. Suppose L is a regular compact frame and let $a, b \in L$ such that $a \vee b = 1$. Since

L is regular, we have $a = \bigvee_{i\in I}\{x_i \in L \mid x_i \prec a\}$ and $b = \bigvee_{i\in I}\{y_i \in L \mid y_i \prec b\}$. Now

$$\begin{aligned} 1 &= a \vee b \\ &= \bigvee_{i\in I}\{x_i \in L \mid x_i \prec a\} \vee \bigvee_{i\in I}\{y_i \in L \mid y_i \prec b\} \\ &= \bigvee_{i\in I}\{x_i \vee y_i \in L \mid x_i \prec a, y_i \prec b\}. \end{aligned}$$

Then $\{x_i \vee y_i \in L \mid x_i \prec a, y_i \prec b\}$ is a cover of L and since L is compact, there exists

$\{x_k \vee y_k \in L \mid k \in K\} \subseteq \{x_i \vee y_i \in L \mid i \in I\}$ where $x_k \prec a$ and $y_k \prec b$ for some finite $K \subseteq I$

such that $\bigvee_{k\in K}(x_k \vee y_k) = 1$ which implies that
$\left(\bigvee_{k\in K}(x_k \vee y_k)\right)^* = 1^* = 0$ implies $\bigwedge_{k\in K} x_k^* \wedge y_k^* = 0$ implies $\left(\bigwedge_{k\in K} x_k^*\right) \wedge \left(\bigwedge_{k\in K} y_k^*\right) = 0$.
Put $u = \bigwedge_{k\in K} x_k^*$ and $v = \bigwedge_{k\in K} y_k^*$, then $u \wedge v = 0$ and

$$\begin{aligned} u \vee a &= \left(\bigwedge_{k\in K} x_k^*\right) \vee a \\ &= \bigwedge_{k\in K}\left(x_k^* \vee a\right) \\ &= \bigwedge_{k\in K}(1) = 1. \end{aligned}$$

Similarly,

$$\begin{aligned} v \vee b &= \left(\bigwedge_{k\in K} y_k^*\right) \vee a \\ &= \bigwedge_{k\in K}\left(y_k^* \vee a\right) \\ &= \bigwedge_{k\in K}(1) = 1. \end{aligned}$$

Hence L is a normal frame. $\square$

In the presence of regularity, normal frames are completely regular just as in the classical case.

Proposition 2.3.1. *[35] A regular normal frame is completely regular.*

Proof. Let L be a regular normal frame and $a \in L$. Take any $x \prec a$, then $x^* \vee a = 1$ and since L is normal there exists $u, v \in L$ such that $u \wedge v = 0$ and $x^* \vee v = 1 = a \vee u$. But $u \wedge v = 0$ implies that $v^* \leq u$, therefore $a \vee v^* \leq a \vee u = 1$, which implies $a \vee v^* = 1$ and hence $v \prec a$. Also, $x^* \vee a = 1$ implies that $x \prec v$. That is, $x \prec v \prec a$. Continuing the process we can interpolate infinitely many elements which are between a and x that are rather below them. Therefore a sequence indexed by rationals between 0 and 1 is formed. Thus, $x \prec\prec a$ as desired. □

Remark 2.3.1. [35] Let L be a regular frame. Then each codense frame homomorphism $h\colon M \to L$ is one-one.

Proposition 2.3.2. *[35] Let L be a compact frame and M a regular frame. Then each dense frame homomorphism $h\colon M \to L$ is one-one.*

Proof. By the previous remark, it suffices to prove that h is codense. Suppose h is dense and let $h(a) = 1$ for some $a \in M$. But $a = \bigvee\{x \in M \mid x \prec a\}$, therefore

$$\begin{aligned} 1 &= h(a) \\ &= h\bigvee\{x \in M \mid x \prec a\} \\ &= \bigvee\{h(x) \mid x \prec a\}. \end{aligned}$$

Which implies that $\{h(x) \mid x \prec a\}$ is a cover of L and since L is compact, there exists

$$x_1,\ x_2,\ \ldots,\ x_n \prec a \text{ such that } \bigvee_{i=1}^{n} h(x_i) = 1.$$

Now, let $u = x_1 \vee x_2 \vee \ldots \vee x_n$, then $u \prec a$ implies $u^* \vee a = 1$, and $h(u) = 1$. Now,

$$h(u^*) \leq (h(u))^* = 0 \text{ enforcing } h(u^*) = 0 \text{ and since } h \text{ is dense, } u^* = 0.$$

Then from $u^* \vee a = 1$, we have $a = 1$. $\square$

Chapter 3

Strict extensions of frames

In this chapter we consider strict extensions and show that dense frame homomorphisms are strict extensions on a regular frame. The notion of strict extensions of frames was introduced by Hong [30]. Strict extensions were further considered by Banaschewski and Hong in terms of filters and general filters, see [9] and [11].

3.1 Strict extensions

In this section, we give the definition of a strict extension and show that dense frame homomorphisms act as examples in regular frames.

Definition 3.1.1. A frame homomorphism $h\colon L \to M$ is called:

i. an *extension* if h is dense and onto.

ii. *strict* if L is generated by $h_*(M)$.

If h satisfy i. and ii., then we say h is a *strict extension* of M.

Now let us look at one of the basic examples of a strict extension of frames.

Example 3.1.1. [9] Consider the frame $\mathcal{D}L$ of all non-empty downsets of a frame L, with union as join and intersection as meet. Define the map $h\colon \mathcal{D}L \to L$ by $h(U) = \bigvee U$.

1. We note that $0_{\mathcal{D}L} = \{0_L\}$ and $1_{\mathcal{D}L} = L$. Therefore

$$h(L) = \bigvee L = 1_L \text{ and } h(\{0\}) = \bigvee \{0\} = 0_L.$$

2. For any $U,\ V \in \mathcal{D}L$, we have

$$\begin{aligned} h(U) \wedge h(V) &= \bigvee U \wedge \bigvee V \\ &= \bigvee (U \cap V) \\ &= h(U \cap V) \\ &= h(U \wedge V) \end{aligned}$$

3. Let $S \subseteq \mathcal{D}L$, then

$$\begin{aligned} h\left(\bigvee S\right) &= h\left(\bigcup_{i\in I} \{U_i \mid U_i \in S\}\right) \\ &= \bigvee \bigcup_{i\in I} \{U_i \mid U_i \in S\} \\ &= \bigcup_{i\in I} \{\bigvee U_i \mid U_i \in S\} \\ &= \bigcup_{i\in I} \{h(U_i) \mid U_i \in S\} \\ &= \bigvee h(S) \end{aligned}$$

hence h is a frame homomorphism with the right adjoint $h_*: L \to \mathcal{D}L$, defined by

$$h_*(a) = \bigvee\{U \in \mathcal{D}L \mid h(U) \leq a\} = \bigvee\{U \in \mathcal{D}L \mid \bigvee U \leq a\} = \downarrow a. \text{ Also,}$$

$$\begin{aligned} h_*(L) &= \bigcup\{\downarrow a \mid a \in L\} \\ &= \mathcal{D}L \text{ since for any } U \in \mathcal{D}L,\ U = \bigvee\{\downarrow a \mid a \in U\} \end{aligned}$$

therefore h is strict. We also observe that $h(U) = 0_L$ implies that $\bigvee U = 0_L$ if and onlyif $U = \{0\}$, and h is dense. Lastly, since for all $x \in L$ there exists $\downarrow x \in \mathcal{D}L$ such that $h(\downarrow x) = \bigvee(\downarrow x) = x$, h is onto. Thus, h is a strict extension of L.

The following lemma is a property stated after Definition 1 in [9].

Lemma 3.1.1. *[9] For a regular frame L, any dense frame homomorphism $h : L \to M$ is strict.*

Proof. Let $a \in L$ and take any $x \prec a$ in L, then $x \leq a$. Since $x \leq h_*h(x)$, we need only to prove that $h_*h(x) \leq a$. We do this by showing that $h_*h(x) \prec a$. Since $x \prec a, x^* \vee a = 1$ and we observe that

$$\begin{aligned} h_*(h(x)) \wedge x^* &= \bigvee\{y \in L \mid h(y) \leq h(x)\} \wedge \bigvee\{z \in L \mid z \wedge x = 0\} \\ &= \bigvee\{y \in L \mid h(y) \leq h(x)\} \wedge \{z \in L \mid z \wedge x = 0\} \\ &= \bigvee\{y \wedge z \in L \mid h(y) \leq h(x),\ z \wedge x = 0\} \end{aligned}$$

Since h is a frame homomorphism, $h(x) \wedge h(z) = h(x \wedge z) = 0$. Now $h(y) \leq h(x)$ implies that $h(y) \wedge h(z) \leq h(x) \wedge h(z) = 0$ which implies that $h(y) \wedge h(z) \leq 0$, and hence $h(y) \wedge h(z) = 0$.

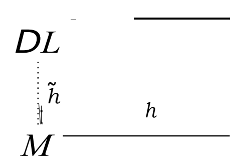
But $0 = h(y) \wedge h(z) = h(y \wedge z)$ and since h is dense, then $y \wedge z = 0$. Therefore

$$
\begin{aligned}
h_*(h(x)) \wedge x^* &= \bigvee \{y \wedge z \in L \mid h(y) \leq h(x),\ z \wedge x = 0\} \\
&= \bigvee \{0\} \\
&= 0.
\end{aligned}
$$

Which implies that x^* is a separating element for $h_*h(x)$ and a, and hence $h_*h(x) \prec a$. Thus for any $x \prec a$, $x \leq h_*h(x) \leq a$ and hence $a = \bigvee \{h_*h(x) \mid h_*h(x) \prec a\}$. $\square$

The following lemma is extracted before Lemma 1 in [9].

Lemma 3.1.2. *[9] For any strict extension $h\colon M \to L$, there exists an onto frame homomorphism $\tilde{h}\colon \mathcal{D}L \to M$ such that $h\tilde{h} = \bigvee$, as in the commutative diagram below*



Proof. Since h is strict, $a = \bigvee \{h_*(x) \mid h_*(x) \leq a\}$ for all $a \in M$. Therefore define $\tilde{h}\colon \mathcal{D}L \to M$ by $\tilde{h}(U) = \bigvee \{h_*(x) \mid x \in U\}$ for any $U \in \mathcal{D}L$. Then for any $a \in L$,

$$
\begin{aligned}
\tilde{h}(\downarrow a) &= \bigvee \{h_*(x) \mid x \in \downarrow a\} \\
&= \bigvee \{h_*(x) \mid x \leq a\} \\
&= h_*(a).
\end{aligned}
$$

We now show that $\tilde{h}$ is a frame homomorphism.

i.

$$\begin{aligned}\widetilde{h}(\downarrow 0) &= h_*(0)\\ &= \bigvee\{x \in L \mid h(x) \leq 0\}\\ &= \bigvee\{x \in L \mid h(x) = 0\}\\ &= \bigvee\{0\} \text{ since } h \text{ is dense, } x = 0.\\ &= 0.\end{aligned}$$

ii. $\widetilde{h}(\downarrow 1) = h_*(1) = \bigvee\{x \in L \mid h(x) \leq 1\} = 1$ since $h(1) \leq 1$.

iii. Let $U, V \in \mathcal{D}L$, then

$$\begin{aligned}\widetilde{h}(U) \wedge \widetilde{h}(V) &= \bigvee\{h_*(x) \mid x \in U\} \wedge \bigvee\{h_*(y) \mid y \in V\}\\ &= \bigvee\{h_*(x) \wedge h_*(y) \mid x \in U,\ y \in V\}\\ &= \bigvee\{h_*(x \wedge y) \mid x \in U, y \in V\}\\ &= \bigvee\{h_*(z) \mid z \in U \cap V\}\\ &= \widetilde{h}(U \cap V).\end{aligned}$$

iv.

$$\begin{aligned}\widetilde{h}\left(\bigcup_{i\in I} U_i\right) &= \bigvee\{h_*(x) \mid x \in \bigcup_{i\in I} U_i\}\\ &= \bigvee_{i\in I}\{h_*(x) \mid x \in U_i\}\\ &= \bigvee_{i\in I}\widetilde{h}(U_i)\end{aligned}$$

Hence $\tilde{h}$ is a frame homomorphism. To show that $\tilde{h}$ is onto, take any $a \in M$, then

$$\begin{aligned} a &= \bigvee \{h_*(x) \mid x \in L\}, \text{ since } h \text{ is strict} \\ &= \bigvee \{h_*(x) \mid h_*(x) \leq a\} \\ &= \bigvee \{\tilde{h}(\downarrow x) \mid \tilde{h}(\downarrow x) \leq a\}, \text{ since } \tilde{h} = h_* \\ &= \tilde{h}\left(\bigvee \{\downarrow x \mid \tilde{h}(\downarrow x) \leq a\}\right), \text{ since } \tilde{h} \text{ is a frame homomorphism.} \end{aligned}$$

Lemma 3.1.3. *[9] If $h: M \to L$ is a strict extension of L such that $h = f \circ g$, where $g: M \to N$ is onto and $f: N \to L$ is any frame homomorphism; then $f_* = g \circ h_*$ and f is a strict extension.*

Proof. We start by proving that $f_* = g \circ h_*$. We show that for $x \in N$ and $y \in L$, $x \leq f_*(y)$ if and only if $x \leq gh_*(y)$. Suppose $x \leq gh_*(y)$, then $f(x) \leq fgh_*(y)$ implies that $f(x) \leq hh_*(y)$ since $fg = h$ and $hh_*(y) = y$ since h is dense and therefore $f(x) \leq y$ which implies that $x \leq f_*(y)$. Conversely, suppose that $x \leq f_*(y)$, then $f(x) \leq y$. Since g is onto, we have $x = gg_*(x)$ therefore $f(gg_*(x)) \leq y$. But $fg = h$ therefore $hg_*(x) \leq y$ implies $g_*(x) \leq h_*(y)$. Then $gg_*(x) \leq gh_*(y)$ and hence $x \leq gh_*(y)$ since $gg_*(x) = x$. We now prove that f is a strict extension. For any $a \in N$, there exists $x \in M$ such that $a = g(x)$ because g is onto. Then since h is strict, then $h_*(L)$ generates M therefore $x = \bigvee \{h_*(y) \mid h_*(y) \leq x$ where $y \in L\}$ and

$$\begin{aligned} a &= g(x) \\ &= g \bigvee \{h_*(y) \mid h_*(y) \leq x\} \\ &= \bigvee \{gh_*(y) \mid h_*(y) \leq x\} \\ &= \bigvee \{f_*(y) \mid h_*(y) \leq x\} \end{aligned}$$

Which implies that $f_*(L)$ generates M and hence f is strict. Now, if $y \in L$ then since h is onto, there exists $x \in M$ such that $h(x) = y$ implies $fg(x) = y$ where $g(x) \in N$ and hence f is onto. To prove that f is dense, suppose $f(a) = 0$ for some $a \in N$. Since g is onto, there exists $x \in M$ such that $a = g(x)$ implies that $fg(x) = 0$ which implies $h(x) = 0$ implies that $x = 0$ implies that $y = g(x) = 0$ and hence f is dense. $\square$

3.2 Filters and strict extensions

Let L be a frame and Q denote a set of filters in L and $P(Q)$ be the power set lattice. Furthermore, we let

$$s_Q L = \{(x, \Sigma) \in L \times P(Q) \mid \text{for any } F \in \Sigma, x \in F\}$$

and let $s : s_Q L \to L$ be the restriction of the first projection to $s_Q L$. Then $s_Q L$ is a subframe of the product frame of L and $P(Q)$ and s is an open dense onto frame homomorphism, which is called the *simple extension* of L with respect to Q, see [[9], [30], and [31]] for details. Let s_* denote the right adjoint of s, then $s_*(x) = (x, \Sigma_x)$ for any $x \in L$, where $\Sigma_x = \{F \in Q \mid x \in F\}$. Clearly $s_*(L)$ is closed under finite meets in s_Q. We present this as a lemma below.

Lemma 3.2.1. *[30] $s_*(L)$ is closed under finite meets in $s_Q L$.*

Proof. Let $x, y \in L$. Then $s_*(x) = (x, \Sigma_x) \in s_Q L$ and $s_*(y) = (y, \Sigma_y) \in s_Q L$. Now

$$\begin{aligned} s_*(x) \wedge s_*(y) &= (x, \Sigma_x) \wedge (y, \Sigma_y) \\ &= (x \wedge y, \Sigma_x \cap \Sigma_y) \\ &= (x \wedge y, \Sigma_{x \wedge y}) \end{aligned}$$

Indeed, if x and y are elements of a filter F, then $x \wedge y \in F$ and hence $s_*(L)$ is closed under finite meets in $s_Q L$. $\square$

Let $t_Q L$ be the subframe of $s_Q L$ generated by $s_*(L)$. Then

$$t_Q L = \{\bigvee \{(x, \Sigma_x) \mid x \in A\} \mid A \subseteq L\}.$$

Let t: $t_Q L \to L$ be the restriction of s to $t_Q L$, which is clearly a dense onto frame homomorphism.

We now define in line with the above notation the following.

Definition 3.2.1. The frame homomorphism t: $t_Q L \to L$ is called the *strict extension* of L with respect to Q.

The next example is given by Banaschewski and Hong in [9].

Example 3.2.1. [9] Let L be a frame, Q a set of filters on L, and $\mathfrak{O}(Q)$ be the topology on Q generated by the sets $Q_a = \{F \in Q \mid a \in F\}$, where $a \in L$. Define the map h : $\mathcal{D}L \to \mathfrak{O}(Q)$ by $h(U) = Q_U = \bigcup_{a \in U} Q_a$, that is, each downset $U \in \mathcal{D}L$ is mapped tothe union of all the filters in Q containing elements of U. Now, $h(U) = \bigcup_{a \in U} Q_a = \{F \in Q \mid F \cap U \neq \emptyset\}$. We show that h is a frame homomorphism. Firstly, we observe that $h(\downarrow 0) = \emptyset$ because $\downarrow 0 \cap F = \emptyset$ for all $F \in Q$ and $h(\downarrow 1) = Q$ since $1 \in F$ for all $F \in Q$.

Now, let $U, V \in \mathcal{D}L$. Then

$$\begin{aligned} h(U \cap V) &= \{F \in Q \mid F \cap (U \cap V) \neq \emptyset\} \\ &= \{F \in Q \mid (F \cap U) \cap (F \cap V) \neq \emptyset\} \\ &= \{F \in Q \mid (F \cap U) \neq \emptyset\} \cap \{F \in Q \mid (F \cap V) \neq \emptyset\} \\ &= h(U) \cap h(V). \end{aligned}$$

And for some $\mathcal{S} \subseteq \mathcal{D}L$, we have

$$
\begin{aligned}
h\left(\bigvee S\right) &= h\left(\bigcup\{U \mid U \in S\}\right)\\
&= \{F \in Q \mid F \cap \bigcup\{U \mid U \in S\} \neq \emptyset\}\\
&= \{F \in Q \mid \bigcup_{U \in S}\{F \cap U \mid U \in S\} \neq \emptyset\}\\
&= \bigcup_{U \in S}\{F \in Q \mid F \cap U \neq \emptyset\}\\
&= \bigcup\{h(U) \mid U \in S\}\\
&= \bigvee h(S).
\end{aligned}
$$

hence h is a frame homomorphism. Now consider the join map $\bigvee : \mathcal{D}L \to L$ which is a frame homomorphism (by Example 3.1.1), then the map $\eta : \mathcal{D}L \to L \times \mathcal{O}(Q)$, $U \mapsto (\bigvee U, Q_{\bigvee U})$, is a frame homomorphism. Now let $\eta(\mathcal{D}L) = \tau_Q L$ and $\tau : \tau_Q L \to L$ be defined by $\tau(\bigvee U, Q_{\bigvee U}) = \bigvee U$. Then the composition $\mathcal{D}L \xrightarrow{\eta} \tau_Q L \xrightarrow{\tau} L$ is a factorisation of $\bigvee$ and hence by Lemma 3.1.3, τ is a strict extension of L with respect to Q with right adjoint $\tau_*(a) = (a, Q_a)$.

3.3 Compactification of frames

Recall that in a frame L, an element $x \in L$ is complemented if there is an element $x' \in L$ such that $x \wedge x' = 0$ and $x \vee x' = 1$. We denote by $C(L)$ the set of all complemented elements in L. A frame L is *zero-dimensional* if $C(L)$ is a base for L. In this section, L is a zero-dimensional frame and Q is the set

$$\{F \mid F \text{ is a non-convergent maximal Boolean filter}\}.$$

By a *maximal Boolean* filter we mean a Boolean filter which is maximal in the collection of Boolean filters in a given frame.

Proposition 3.3.1. *[30] For a zero-dimensional frame L, the following are equivalent:*

1. *L is compact;*
2. *Every Boolean filter in L is clustered;*
3. *Every maximal Boolean filter in L is convergent.*

Proof. (1) ⇒ (2): It is immediate from Corollary 2.2.1.

(2) ⇒ (3): We note that a Boolean filter in L is maximal if and only if $sec\,F \cap C(L) \subseteq F$. Thus the implication follows from 5) of Remark 1.2 in [30], for $C(L)$ is a base for L.

(3) ⇒ (1): Suppose that there is a cover C of L which does not have a finite subcover. Let $T = \{t \in C(L) \mid t \leq c$ for some $c \in C\}$, then T is a cover of L, which does not have a finite subcover. Thus $\{x' \mid x \in T\}$ generates a Boolean filter, which is denoted by F. Let G be a maximal Boolean filter containing F. By the assumption, G is convergent, so $G \cap T \neq \emptyset$. Pick $t \in G \cap T$, then $t, t' \in G$ which is a contradiction. □

Here we show that t_QL is a zero-dimensional compact frame and hence $t: t_QL \to L$ is a zero-dimensional compactification of L. Before presenting this result, we need a series of lemmas.

Lemma 3.3.1. *[16] In any zero-dimensional frame, every completely prime filter is Boolean.*

Proof. Suppose that L is a zero-dimensional frame and P is a completely prime filter in L with $x \in P$. Then $x = \bigvee T$ for some $T \subseteq C(L)$. Since P is completely prime, $t \in P$ for some $t \in T$. Thus $t \in P \cap C(L)$ with $t \leq x$. Hence, P is Boolean. □

Lemma 3.3.2. *If $a, b \in C(L)$, then $a \wedge b \in C(L)$.*

Proof. Let $a, b \in C(L)$. Then there exists $a', b' \in L$ such that $a \wedge a' = 0$ and $a \vee a' = 1$, also $b \wedge b' = 0$ and $b \vee b' = 1$. Now, we claim that $a' \vee b'$ is a complement for $a \wedge b$.

$$\begin{aligned}(a \wedge b) \wedge (a' \vee b') &= (a \wedge b \wedge a') \vee (a \wedge b \wedge b') \\ &= 0 \vee 0 \\ &= 0.\end{aligned}$$

Now let $a' \vee b' = q$, therefore

$$\begin{aligned}(a \wedge b) \vee q &= (a \vee q) \wedge (b \vee q) \\ &= (a \vee a' \vee b') \wedge (b \vee a' \vee b') \\ &= 1 \wedge 1 \\ &= 1.\end{aligned}$$

$\square$

Lemma 3.3.3. *[30] $s_*\big(C(L)\big)$ is contained in $C(t_Q L)$ and is closed under finite meets in $t_Q L$. Furthermore, $s_*\big(C(L)\big)$ generates $t_Q L$.*

Proof. Let $(x, \ {}_x) \in s_*\big(C(L)\big)$. Then $(x, \ {}_x) \in t_Q L$ and $x \in C(L)$. Since ${}_x = \{F \in Q \mid x \in F\}$ and F is a maximal Boolean filter, it follows that either $x \in F$ or $x' \in F$. Since $x \in C(L)$, so $x' \in C(L)$. If $x \in F$, then $x' \not\in F$ but $x' \in G$ where G is a maximal

Boolean filter in Q. In other words $(x', \ _{x}) \in t_Q L$, therefore

$$
\begin{aligned}
\big(x, \ _{x}\big) \wedge \big(x', \ _{x}\big) &= \big(x \wedge x', \ _{x} \cap \ _{x}\big) \\
&= \big(x \wedge x', \ _{x \wedge x}\big) \\
&= \big(0, \ _{0}\big) \\
&= (0, \emptyset)
\end{aligned}
$$

and

$$
\begin{aligned}
\big(x, \ _{x}\big) \vee \big(x', \ _{x}\big) &= \big(x \vee x', \ _{x} \cup \ _{x}\big) \\
&= \big(x \vee x', \ _{x \vee x}\big) \\
&= \big(1, \ _{1}\big) \\
&= (1, Q).
\end{aligned}
$$

Hence $\big(x, \ _{x}\big) \in C\big(t_Q L\big)$, which implies that $s_*\big(C(L)\big) \subseteq C(t_Q L)$. Now let

$$(x, \ _{x}), (y, \ _{y}) \in s_*\big(C(L)\big), \text{ therefore } x, y \in C(L)$$

and by Lemma 3.3.2, $x \wedge y \in C(L)$. Then

$$
\begin{aligned}
\big(x \wedge y, \ _{x \wedge y}\big) &= \big(x \wedge y, \ _{x} \cap \ _{y}\big) \\
&= \big(x, \ _{x}\big) \wedge \big(y, \ _{y}\big) \in s_*\big(C(L)\big)
\end{aligned}
$$

and hence $s_*\big(C(L)\big)$ is closed under finite meets. To show that $s_*\big(C(L)\big)$ generates $t_Q L$, note that for any $a \in L$, $a = \bigvee\{x \in L \mid x \in C(L)\}$ since L is zero-dimensional and since Q consists of maximal Boolean filters, we have $a \in F \in Q$ implies there exists $x \in C(L)$ such that $x \leq a$ which implies that $x \in C(L) \cap \downarrow a$. Therefore $(a, {}_{a}) = \bigvee\{(x, {}_{x}) \mid x \in C(L) \cap \downarrow a\}$, thus $t_Q L$ is generated by $s_*\big(C(L)\big)$. □

Lemma 3.3.4. *[30] For any maximal Boolean filter Ψ in $t_Q L$, $t(\Psi)$ is also a maximal Boolean filter in L.*

Proof. Let $x \in t(\Psi)$. Since t is onto, there exists $(x, {}_{x}) \in \Psi$ such that $t(x, {}_{x}) = x$. Since Ψ is a maximal Boolean filter, it follows that there is a complemented $(y, {}_{y}) \in \Psi$ such that $(y, {}_{y}) \leq (x, {}_{x})$. Therefore there is $(z, {}_{z}) \in C(t_Q L)$ such that

$$
\begin{aligned}
(y, {}_{y}) \wedge (z, {}_{z}) &= (y \wedge z, {}_{y\wedge z}) \\
&= (0, {}_{0}) \\
&= (0, \emptyset)
\end{aligned}
$$

and

$$
\begin{aligned}
(y, {}_{y}) \vee (z, {}_{z}) &= (y \vee z, {}_{y\vee z}) \\
&= (1, {}_{1}) \\
&= (1, Q).
\end{aligned}
$$

Now, $t(y, {}_{y}) = y$ and $t(z, {}_{z}) = z$. Furthermore, $(y, {}_{y}) \leq (x, {}_{x})$ implies $y \leq x$. Also, $y \wedge z = 0$ and $y \vee z = 1$ shows that y is complemented in L and $y \in t(\Psi)$. Hence $t(\Psi)$ is a Boolean filter in L. To show that $t(\Psi)$ is maximal, note the following:

$t(\Psi) = \{x \in L \mid (x, {}_{x}) \in \Psi\}$ and $sec\ t(\Psi) = \{y \in L \mid \text{ for all } x \in t(\Psi), x \wedge y \neq 0\}$.

Therefore

$$
\begin{aligned}
s_*\big(sec\ t(\Psi)\big) &= \{s_*(y) \mid y \in sec\ t(\Psi)\} \\
&= \{\big(y, \ _y\big) \in t_Q L \mid \text{ for all } x \in t(\Psi), x \wedge y \neq 0\} \\
&= \{\big(y, \ _y\big) \in t_Q L \mid \text{ for all } \big(x, \ _x\big) \in \Psi, \big(x \wedge y, \ _{x\wedge y}\big) \neq (0, \emptyset)\} \\
&= sec\ \Psi.
\end{aligned}
$$

Now

$$
\begin{aligned}
s_*\big(sec\ t(\Psi) \cap C(L)\big) &= s_*\big(sec\ t(\Psi)\big) \cap s_*\big(C(L)\big) \\
&\subseteq s_*\big(sec\ t(\Psi)\big) \cap C\big(t_Q L\big) \\
&= sec\ \Psi \cap C\big(t_Q L\big)
\end{aligned}
$$

then we have

$$
s_*\big(sec\ t(\Psi) \cap C(L)\big) \subseteq sec\ \Psi \cap C\big(t_Q L\big) \subseteq \Psi
$$

since Ψ is a maximal Boolean filter, and hence $sec\ t(\Psi) \cap C(L) \subseteq t(\Psi)$ which implies that $t(\Psi)$ is a maximal Boolean filter. $\square$

Theorem 3.3.1. *[30] $t_Q L$ is a zero-dimensional compact frame and hence $t\colon t_Q L \to L$ is a zero-dimensional compactification of L.*

Proof. Take any maximal Boolean filter Ψ in $t_Q L$ and suppose $t(\Psi)$ is convergent in L. Let

$$
\Phi = \big\{(x, \ _x) \in t_Q L \mid (y, \ _y) \leq (x, \ _x) \text{ for some } (y, \ _y) \in t^{-1}(t(\Psi)) \cap C(t_Q L)'\big\},
$$

then Φ is a Boolean filter in $t_Q L$ containing Ψ, hence $\Phi = \Psi$. By Proposition 2.2.4, $t^{-1}(t(\Psi))$ is convergent and since $t_Q L$ is a zero-dimensional frame, $\Psi = \Phi$ is also convergent in $t_Q L$. Now suppose that $t(\Psi)$ is not convergent, then $t(\Psi) \in Q$. Take any cover S of $t_Q L$ with $S \subseteq s_*\big(C(L)$ which generates $t_Q L$. Let $p : t_Q L \to P(Q)$, defined by $p((x, \ {}_x)) = \ {}_x$. Then p is a frame homomorphism, hence $p(S)$ is a cover of $P(Q)$; $t(\Psi) \in p((x, \ {}_x)) = \ {}_x$ for some $(x, \ {}_x) \in S$. Hence there is $(y, \ {}_y) \in \Psi$ with $t(y, \ {}_y) = x$ so that

$$(y, \ {}_y) \leq t_*(x) = (x, \ {}_x)$$

implies $(x, \ {}_x) \in S \cap \Psi$, and therefore Ψ is convergent in $t_Q L$. Thus by Proposition 3.3.1, $t_Q L$ is compact. □

3.4 Some constructive results

In this section we give a constructive compactification of completely regular frames. These results will show that the category **RegKFrm** of regular compact frames is a coreflection of the category **CRegFrm** of completely regular frames.

Proposition 3.4.1. *[35] Let L be a frame and define the following maps $v : I(L) \to L$ and $\alpha : L \to I(L)$ by $I \to \bigvee I$ and $a \to \downarrow a$, respectively, for all $I \in I(L)$ and for all $a \in L$. Then v is a dense onto frame homomorphism.*

Proof. We observe that for all $I \in I(L)$ and for all $a \in L$ we have

$$\begin{aligned} v(\alpha(a)) &= v(\downarrow a) \\ &= \bigvee(\downarrow a) \\ &= a \end{aligned}$$

and

$$\alpha(v(I)) = \alpha\left(\bigvee I\right) = \downarrow\left(\bigvee I\right) \supseteq I.$$

Therefore v is a left Galois adjoint and hence preserves arbitrary joins. Now, for any $I,\ J \in \mathcal{I}(L)$ we have

$$\begin{aligned} v(I) \wedge v(J) &= \left(\bigvee I\right) \wedge \left(\bigvee J\right) \\ &= \bigvee \{a \wedge b \mid a \in I \text{ and } b \in J\} \\ &\leq \bigvee \{c \mid c \in I \cap J\} \\ &= v(I \cap J). \end{aligned}$$

But $v(I \cap J) \leq v(I) \cap v(J)$ since $\bigvee I \cap J \leq \bigvee I \wedge \bigvee J$ and hence $v(I) \wedge v(J) = v(I \cap J)$. The map v preserves the top element since $v(L) = \bigvee L = 1$. For any $a \in L$, there exists $\downarrow a \in \mathcal{I}(L)$ such that $v(\alpha(a)) = v(\downarrow a) = \bigvee(\downarrow a) = a$, hence v is onto. To show that v is dense, suppose $v(I) = 0$ for some $I \in \mathcal{I}(L)$. Then $\bigvee I = 0$ implies $I = \{0\}$. Hence v isa dense onto frame homomorphism. $\square$

Definition 3.4.1. [35] An ideal I in a frame L is *regular* if for every $a \in I$, there exists $b \in I$ such that $a \prec\prec b$.

The following propositions are crucial in the sense that they play a vital role in Stone-Čech compactification of completely regular frames.

Proposition 3.4.2. *[35] The set of regular ideals in a frame L, denoted RL, is a subframe of $\mathcal{I}(L)$. Hence it is a compact frame.*

Proof. Let $I, J \in RL$ and $a \in I \cap J$. This means that $a \in I$ and $a \in J$ but both I and J are regular, so there exists $b_1 \in I$ and $b_2 \in J$ such that $a \prec\prec b_1$ and $a \prec\prec b_2$ hence $a = a \wedge a \prec\prec b_1 \wedge b_2 \in I \cap J$. Now let $(I_a)_{a \in \Lambda} \in RL$ and $F = \bigcup_{k \in K} I_k$ for some finite $K \subseteq \Lambda$. For each $a_k \in I_k$ there exists $b_k \in I_k$ such that $a_k \prec\prec b_k$ for all $k \in K$. Thus we have $\bigvee_{k \in K} a_k \prec\prec \bigvee_{k \in K} b_k \in F$. Showing that RL is a subframe of $\mathcal{I}(L)$ and hence compact. $\square$

Proposition 3.4.3. *[35] Let L be a frame. Then for each $a \in L$, the set*

$$\sigma(a) = \{b \in L \mid b \prec\prec a\} \text{ is a regular ideal.}$$

Proof. To show that $\sigma(a)$ is an ideal, let $b_1, b_2 \in \sigma(a)$. Therefore $b_1, b_2 \prec\prec a$ implies that $b_1 \vee b_2 \prec\prec a$ and hence $b_1 \vee b_2 \in \sigma(a)$. Now, suppose that $b_1 \leq b_2$ and $b_2 \in \sigma(a)$ for some $b_1 \in L$. Then $b_1 \leq b_2 \prec\prec a$ implies $b_1 \prec\prec a$ which implies $b_1 \in \sigma(a)$ and hence $\sigma(a)$ is an ideal. $\sigma(a)$ is regular since the relation $\prec\prec$ is interpolative. $\square$

Lemma 3.4.1. *[35] If $b \prec\prec a$ in a frame L, then $\sigma(b) \prec \sigma(a)$ in the frame RL.*

Proof. Suppose $b \prec\prec a$ in the frame L. Since the relation $\prec\prec$ is interpolative, there exists $x, y \in L$ such that $b \prec\prec x \prec y \prec\prec a$. Since $b^* \wedge b = 0$, we have $\sigma(b^*) \cap \sigma(b) = \{0\}$ and $b \prec\prec x$ implies $x^* \prec\prec b^*$ which implies that $x^* \in \sigma(b^*) \subseteq \sigma(b)^*$ therefore $\sigma(b)^* \vee \sigma(a) = 1_{RL} = L$ implies $\sigma(b) \prec \sigma(a)$. $\square$

Proposition 3.4.4. *[35] If L is a completely regular compact frame, then $v: RL \to L, I \to \bigvee I$ and $\sigma: L \to RL,\ a \to \{b \in L \mid b \prec\prec a\}$ are mutually inverse isomorphisms.*

Proof. Let $I \in RL$ and $a \in I$. Since L is completely regular,

$$a = \bigvee\{b \in L \mid b \prec\prec a\} \prec\prec \bigvee I \text{ since } I \text{ is regular.}$$

This implies that

$a \in \{b \in L \mid b \prec\prec \bigvee I\}$ implies $a \in \{b \in L \mid b \prec\prec v(I)\}$ which implies $a \in \sigma\big(v(I)\big)$ and hence $I \subseteq \sigma\big(v(I)\big)$.

On the other hand, if $a \in \sigma\big(v(I)\big)$ then $a \prec \bigvee I$ and $a^* \vee (\bigvee I) = 1$. Since L is compact, there exists $x_k \in I$ where $k \in K$ for some finite K such that

$$a^* \vee \Big(\bigvee_{k \in K} x_k\Big) = 1. \text{ Let } x = \bigvee_{k \in K} x_k \in I \text{ and hence } \sigma\big(v(I)\big) \subseteq I.$$

$\square$

3.5 Compactifications of completely regular frames

The following lemma is a property stated in [35], page 133 (**4.4**).

Lemma 3.5.1. *Let $h \colon L \to M$ be a frame homomorphism. If $I \subseteq L$ is a regular ideal, then the following conditions are satisfied:*

1. *$a \vee b \in h(I)$ for all $a, b \in h(I)$; and*
2. *for each $a \in h(I)$, there exists $b \in h(I)$ such that $a \prec\prec b$.*

Proof. 1. Let $a, b \in h(I)$, then there exists $x, y \in I$ such that $a = h(x)$ and $b = h(y)$.

Therefore $a \vee b = h(x) \vee h(y) = h(x \vee y)$ implies $a \vee b \in h(I)$ since $x \vee y \in I$.

2. Let $a \in h(I)$, then there exists $x \in I$ such that $a = h(x)$. Since I is regular, there exists $y \in I$ such that $x \prec\prec y$. Put $b = h(y)$, and hence $a \prec\prec b$.

$\square$

Theorem 3.5.1. *[35] Let L be a completely regular frame and define $v_L\colon RL \to L$ by $v_L(I) = \bigvee I$ and $v_M : RM \to M$ by $v_M(J) = \bigvee J$. If M is a compact frame and $h : L \to M$ is a frame homomorphism, then there exists a functor*

$$R\colon \mathbf{CRegFrm} \to \mathbf{KRegFrm}$$

such that the following diagram commutes:

$$\begin{array}{ccc} RL & \xrightarrow{Rh} & RM \\ {\scriptstyle v_L}\big\downarrow & & \big\downarrow{\scriptstyle v_M} \\ L & \xrightarrow[h]{} & M \end{array}$$

Proof. Consider the categories **CRegFrm** and **KRegFrm** of completely regular frames and compact regular frames, respectively, and define $R\colon$ **CRegFrm** $\to$ **KRegFrm** by

$$L \to RL \text{ for all } L \in \mathbf{CRegFrm} \text{ and } R(h\colon L \to M) = Rh\colon RL \to RM,\ h(I) \to {\downarrow} h(I)$$

for any frame homomorphism $h\colon L \to M$ in **CRegFrm**. Firstly note that, in line with the previous lemma, $Rh(I) = {\downarrow} h(I)$ extends $h(I)$ into a regular ideal for each regular ideal I in L. Next we prove that R is a functor. Let $h : A \to B$ and $g : B \to C$ be frame homomorphisms in **CRegFrm**, then

$$\begin{aligned}(Rg) \circ (Rh)(I) &= Rg\big({\downarrow} h(I)\big) \\ &= {\downarrow} g\big({\downarrow} h(I)\big) \\ &= {\downarrow} gh(I) \\ &= R(gh)(I).\end{aligned}$$

Now take any $A \in$ **CRegFrm** and its identity morphism $id_A\colon A \to A,\ a \mapsto a$. Then

$$Rid_A\colon RA \to RA,\ Rid_A(I) = {\downarrow} id_A(I) = {\downarrow} I = I \text{ and hence } R \text{ is a functor.}$$

Lastly, $v_M \circ Rh(I) = v_M({\downarrow} h(I)) = \bigvee {\downarrow} h(I) = \bigvee h(I)$ and $h \circ v_L(I) = h(\bigvee I) = \bigvee h(I)$. An

$$\begin{aligned} v_M \circ (Rh \circ \sigma_L)(a) &= v_M \circ \left(Rh \circ \sigma_L(a)\right) \\ &= v_M\left({\downarrow} h\left(\sigma_L(a)\right)\right) \\ &= \bigvee {\downarrow} h\left(\sigma_L(a)\right) \\ &= \bigvee h\left(\sigma_L(a)\right) \\ &= h\left(\bigvee \sigma_L(a)\right) \\ &= h(a). \end{aligned}$$

$\square$

Hence the category **RegKFrm** is coreflective in **CRegFrm**.

Chapter 4

General filters on frames

General filters on frames were introduced by Banaschewski to address the deficiency of classical filters in frames to describe the notion of completeness. In [16] Bhattacharjee and Naidoo revisited the notion of general filters in a frame and introduced the concepts of clustering of general filters, maximal general filters and general ultrafilters. Using these concepts the authors then characterised almost compact frames and Boolean frames.

4.1 Convergence of general filters on frames

To address the deficiency of classical filters in frames on the notion of compactness, Banaschewski [7] introduced general filters on a frame as follows. Let L be a frame, and denote by $\mathbf{2} = \{0, 1\}$ the two-element frame. The characteristic function of a set $K \subseteq L$ is the mapping $\chi_K : L \to \mathbf{2}$ defined by $\chi_K(a) = \begin{cases} 1 & \text{if } a \in K, \\ 0 & \text{if } a \notin K. \end{cases}$

A subset $F \subseteq L$ is a filter in a frame L if and only if χ_F is a meet-semilattice homomorphism. Replacing $\mathbf{2}$ with an arbitrary frame T, Banaschewski defined a $T-$*valued filter* ona frame L to be a $0-$meet semilattice homomorphism $\psi : L \to T$. If T is unspecified, one speaks of a general filter on L.

Definition 4.1.1. [16] A *general filter* on a frame L is any bounded meet semilattice homomorphism $\phi\colon L \to T$ from L into any arbitrary frame T.

Definition 4.1.2. [16] Let $\phi\colon L \to T$ be a general filter. We say ϕ is:

1. *prime* if ϕ is a lattice homomorphism;
2. *completely prime* if ϕ preserves all joins if and only if ϕ is a frame homomorphism;
3. *regular* if $\phi(a) = \bigvee\{\phi(x) \mid x \prec a\}$ for all $a \in L$;
4. *convergent* if ϕ takes covers to covers;
5. *strongly convergent* if $h \leq \phi$ for some frame homomorphism $h\colon L \to T$.

Proposition 4.1.1. *[16] A convergent regular (classical) filter is completely prime.*

Proof. Let F be a convergent regular filter in a frame L, and $\bigvee S \in F$ for some $S \subseteq L$. Then there exists $y \in F$ such that $y \prec \bigvee S$. The set $\{y^*\} \cup S$ is a cover of L, which implies that $F \cap (\{y^*\} \cup S) \neq \emptyset$. Now, since $y^* \notin F$, we must have $F \cap S \neq \emptyset$, and hence F is completely prime. $\square$

Proposition 4.1.2. *[16] A convergent regular general filter is a frame homomorphism.*

Proof. Let $\phi : L \to T$ be a convergent regular general filter, we show that ϕ preserves all joins. For any $A \subseteq L$, we have $\phi(\bigvee A) \geq \bigvee\{\phi(a) \mid a \in A\}$. Since ϕ is regular,

$$\phi(\bigvee A) \geq \bigvee\{\phi(x) \mid x \prec \bigvee A\}.$$ For any $x \prec \bigvee A$, $\{x^*\} \cup A$ is a cover for L,

and since ϕ is convergent, we have $\phi(x^*) \vee \bigvee\{\phi(a) \mid a \in A\} = 1$ which implies that $\phi(x) \leq \bigvee\{\phi(a) \mid a \in A\}$ since $\phi(x) \wedge \phi(x^*) = 0$. Thus $\phi(\bigvee A) = \bigvee\{\phi(x) \mid x \prec \bigvee A\} \leq \bigvee\{\phi(a) \mid a \in A\}$ which implies that

$$\phi(\bigvee A) \leq \bigvee\{\phi(a) \mid a \in A\} \text{ hence } \phi(\bigvee A) = \bigvee\{\phi(a) \mid a \in A\}.$$

□

Recall the following characterization of clustering in terms of covers : A classical filter F in a frame L is clustered if and only if for every cover C of L there exists $c \in C$ such that $c^* \not\in F$. Given a general filter $\phi: L \to T$, we denote by $\phi^{\leftarrow}(1)$ the classical filter

$$\phi^{\leftarrow}(1) = \{x \in L \mid \phi(x) = 1\}.$$

The following lemma shows that the convergence of classical filters is stronger than the convergence of general filters.

Lemma 4.1.1. *[16] Given a general filter $\phi: L \to T$, if $\phi^{\leftarrow}(1)$ is convergent, then ϕ is also convergent.*

Proof. Suppose the classical filter $\phi^{\leftarrow}(1)$ is convergent and let C be a cover of L, then $\phi^{\leftarrow}(1) \cap C \neq \emptyset$. Say $a \in \phi^{\leftarrow}(1) \cap C$ implies $a \in \phi^{\leftarrow}(1)$ and $a \in C$ with $\phi(a) = 1$. Hence $\bigvee \phi(C) = 1$ and ϕ is convergent. □

The converse is however not true, for, the identity map id_L: $\mathbf{4} \to \mathbf{4}$ ($\mathbf{4} = \{0, a, a^*, 1\}$), which is a convergent general filter. The subset $\{a, a^*\}$ is a cover of $\mathbf{4}$ so $id_L(\{a, a^*\}) = \{a, a^*\}$ which is still a cover for $\mathbf{4}$. But $\{a, a^*\} \cap id_L^{\leftarrow}(1) = \emptyset$ since $id_L(a) = a$ 1 and $id_L(a^*) = a^* \neq 1$, hence $id_L^{\leftarrow}(1)$ is not convergent.

Definition 4.1.3. [16] A general filter $\phi: L \to T$ is said to be clustered if for every cover C of L, there exists $c \in C$ such that $\phi(c^*) \neq 1$.

We note that if $\phi^{\leftarrow}(1)$ clusters, then for any cover C of L, there exists $c \in C$ with $c^* \not\in \phi^{\leftarrow}(1)$ implies $\phi(c^*) \neq 1$. Hence $\phi^{\leftarrow}(1)$ clustered implies ϕ clusters.

Proposition 4.1.3. *[16] A convergent general filter clusters.*

Proof. Let $\phi : L \to T$ be a convergent general filter. If ϕ is not clustered, then there exists a cover C of L such that $\phi(c^*) = 1$ for every $c \in C$. Since ϕ is a semilattice homomorphism, we have $0 = \phi(c) \wedge \phi(c^*) = \phi(c)$ for every $c \in C$, which implies that $\phi(C)$ is not a cover of T, contrary to the hypothesis that ϕ is convergent. $\square$

We show that almost compactness can be characterised in terms of general filters. Hong [30] characterized almost compactness in terms of maximal filters in a frame L and compactness in terms of maximal filters in a regular frame L. We have included the results as Corollary 2.2.1 and Corollary 2.2.2. We start with the following lemma.

Lemma 4.1.2. *[16] Every general filter on a frame is below a maximal one.*

Proof. Let $\phi : L \to T$ be a general filter on L, and $S = \{\mu : L \to T \mid \phi \leq \mu\}$. Let $C \subseteq S$ be a chain, and define a map $\psi : L \to T$ by $\psi(a) = \bigvee\{\gamma(a) \mid \gamma \in C\}$. We show that ψ is a general filter on L. It is obvious that $\psi(0) = 0$ and $\psi(1) = 1$. To show that ψ preserves binary meets, take any $a, b \in L$, then

$$\begin{aligned}\psi(a) \wedge \psi(b) &= \bigvee\{\gamma(a) \mid \gamma \in C\} \wedge \bigvee\{\mu(b) \mid \mu \in C\}\\ &= \bigvee(\{\gamma(a) \mid \gamma \in C\} \wedge \{\mu(b) \mid \mu \in C\})\\ &= \bigvee\{\gamma(a) \wedge \mu(b) \mid \gamma, \mu \in C\}\end{aligned}$$

since C is a chain, for any $\gamma, \mu \in C$,

$$\text{either } \gamma(a) \wedge \mu(a) \leq \gamma(a) \wedge \gamma(b) = \gamma(a \wedge b) \text{ or } \gamma(a) \wedge \mu(a) \leq \mu(a) \wedge \mu(b) = \mu(a \wedge b).$$

$$\begin{aligned}\psi(a) \wedge \psi(b) &= \bigvee\{\gamma(a) \wedge \mu(b) \mid \gamma, \mu \in C\}\\ &\leq \bigvee\{\delta(a \wedge b) \mid \delta \in C\}\\ &= \psi(a \wedge b)\end{aligned}$$

and hence $\psi(a) \wedge \psi(b) = \psi(a \wedge b)$. Thus, ψ is an upper bound for C, and by Zorn's lemma S has a maximal element. $\square$

Observation 4.1.1. *([16], after Lemma 2.4) A general filter that is below a clustered one is clustered. To see this, suppose that $\psi \leq \phi$ and ϕ is clustered. Then there exists $c \in C$ such that $\phi(c^*) \neq 1$. So indeed $\psi(c^*) \leq \phi(c^*) \neq 1$. Hence $\psi(c^*) \neq 1$.*

Proposition 4.1.4. *[16] The following are equivalent in a frame L.*

1. *L is almost compact.*

2. *Every general filter on L clusters.*

3. *Every maximal general filter on L clusters.*

Proof. (1) $\Rightarrow$ (2): Suppose that L is almost compact and let $\phi : L \to T$ be a general filter on L. Let C be a cover of L, then since L is almost compact there exists a finite $\{c_1, c_2, \ldots, c_n\} \subseteq C$ such that $(c_1 \vee c_2 \vee \ldots, \vee c_n)^* = 0$. Then $c_1^* \wedge c_2^* \wedge \ldots, \wedge c_n^* = 0$, and hence

$$0 = \phi(c_1^* \wedge c_2^* \wedge \ldots, \wedge c_n^*) = \phi(c_1^*) \wedge \phi(c_2^*) \wedge, \ldots, \wedge \phi(c_n^*)$$

since ϕ is a meet-semilattice homomorphism. If $\phi(c_i^*) = 1$ for all $i = 1, 2, \ldots, n$, then this contradicts the previous equation. Thus for some $i = 1, 2, \ldots, n$, $\phi(c_i^*)$ 1 and hence ϕ clusters.

(2) $\Rightarrow$ (3): This is trivial.

(3) $\Rightarrow$ (1): Let $F \subseteq L$ be a classical filter, and consider the general filter $\chi_F : L \to \mathbf{2}$. By Lemma 4.1.2, there is a maximal filter $\tau : L \to \mathbf{2}$ with $\chi_F \leq \tau$. By the present hypothesis, τ clusters, which implies χ_F clusters, and hence F clusters. It therefore follows from Corollary 2.2.1 that L is almost compact. $\square$

We give the following lemma without proof.

Lemma 4.1.3. *[16] A classical filter F in a frame L is an ultrafilter if and only if for any $x \in L$, either $x \in F$ or $x^* \in F$.*

Definition 4.1.4. [16] We say a general filter $\phi : L \to T$ is:

1. an *ultrafilter* if for all $x \in L$, $\phi(x) \vee \phi(x^*) = 1$;
2. *balanced* if $\phi(x) \neq 0$ for any $x \in D(L)$.

The following proposition and its proof is found in [18], here we leave out the proof.

Proposition 4.1.5. *[18] Every filter in a frame is the intersection of prime filters containing it.*

The proposition below is culled in [16] and the proof is found in [18].

Proposition 4.1.6. *[16] A classical filter F is an ultrafilter if and only if it is prime and balanced.*

In accordance with the above Proposition 4.1.6, the necessary part also holds in the case of general filters.

Theorem 4.1.1. *[16] Every general ultrafilter is prime and balanced.*

Proof. Let $\phi : L \to T$ be a general ultrafilter. To show that ϕ is balanced, let $x \in L$ be dense, then $x^* = 0$ and $\phi(x^*) = 0$. Since ϕ is an ultrafilter, $1 = \phi(x) \vee \phi(x^*) = \phi(x)$ implies $\phi(x) \neq 0$ and hence ϕ is balanced. Now, to show that ϕ is prime let $a, b \in L$. Since ϕ is an ultrafilter, we have $\phi(z) \vee \phi(z^*) = 1$. Therefore

$$\begin{aligned}
\phi(a \vee b) &= \phi(a \vee b) \wedge \big[\phi(a) \vee \phi(a^*)\big] \wedge \big[\phi(b) \vee \phi(b^*)\big] \\
&= \phi(a \vee b) \wedge \big[\phi(a) \vee \phi(a^*)\big] \wedge \phi(a \vee b) \wedge \big[\phi(b) \vee \phi(b^*)\big] \\
&= \big[\phi(a \vee b) \wedge \phi(a) \vee \phi(a \vee b) \wedge \phi(a^*)\big] \wedge \big[\phi(a \vee b) \wedge \phi(b) \vee \phi(a \vee b) \wedge \phi(b^*)\big] \\
&= \big[\phi\big((a \vee b) \wedge a\big) \vee \phi\big((a \vee b) \wedge a^*\big)\big] \wedge \big[\phi\big((a \vee b) \wedge b\big) \vee \phi\big((a \vee b) \wedge b^*\big)\big] \\
&= \big[\phi(a) \vee \phi(b \wedge a^*)\big] \wedge \big[\phi(b) \vee \phi(a \wedge b^*)\big] \\
&\leq \big[\phi(a) \vee \phi(b)\big] \wedge \big[\phi(b) \vee \phi(a)\big] \\
&= \phi(a) \vee \phi(b),
\end{aligned}$$

whence we deduce that $\phi(a) \vee \phi(b) = \phi(a \vee b)$ as the opposite inequality holds by virtue of ϕ being an increasing map. Therefore ϕ is a prime filter. $\square$

In [16] an example is given which shows that the converse of the above theorem is not true.

Proposition 4.1.7. *[16] Every general ultrafilter is maximal.*

Proof. Let $\phi\colon L \to T$ be a general ultrafilter on L, and consider any general filter $\tau\colon L \to T$ with $\phi \leq \tau$. For any $x \in L$, we have $\phi(x^*) \leq \tau(x^*)$. Since ϕ is an ultrafilter,

$$1 = \phi(x) \vee \phi(x^*) \leq \phi(x) \vee \tau(x^*)$$

which implies that $\tau(x) \leq \phi(x)$ since $\tau(x) \wedge \tau(x^*) = 0$. Thus $\tau \leq \phi$ implies $\phi = \tau$. $\square$

Combining Proposition 4.1.7 and Proposition 4.1.4, the following corollary is immediate.

Corollary 4.1.1. *[16] A frame is almost compact if and only if every general ultrafilter on it clusters.*

Definition 4.1.5. [16] A general filter $\phi\colon L \to T$ on a frame L is Boolean if for every $a \in L$ with $\phi(a) = 1$, there is a complemented element $c \in L$ with $c \leq a$ such that $\phi(c) \neq 0$.

Lemma 4.1.4. *[16]On a zero-dimensional frame, every completely prime general filter is Boolean.*

Proof. Let $\phi : L \to T$ be a completely prime general filter on a zero-dimensional frame L. Suppose that $x \in L$ and $\phi(x) = 1$. Since L is zero-dimensional, there exists $A \subseteq C(L)$ such that $x = \bigvee A$, and since ϕ is completely prime, we have $1 = \phi(\bigvee A) = \bigvee\{\phi(a) \mid a \in A\}$. Hence $\phi(a) \neq 0$ for some $a \in A$, and $a \leq x$ since $a \in C(L)$. $\square$

Theorem 4.1.2. *[16] The following are equivalent for a frame L:*

1. *L is Boolean.*
2. *Every general filter on L is regular.*
3. *The general filter $b_L : L \to BL$ is regular.*
4. *Every general filter in L is Boolean.*
5. *Every general prime filter on L is an ultrafilter.*

Proof. (1) ⇒ (2): Assume that L is Boolean, and let $\phi : L \to T$ be a general filter on L. For any $a \in L$, $a \prec a$, so that $\phi(a) = \bigvee\{\phi(x) \mid x \prec a\}$, and hence ϕ is regular.

(2) ⇒ (3): Trivial.

(3) ⇒ (4): Assume that $b_L : L \to BL$ is regular. Then, for any $a \in L$ we have

$$\begin{aligned} a &\leq a^{**} \\ &= b_L(a) \\ &= \bigvee\{b_L(x) \mid x \prec a\} \\ &= \bigvee\{x^{**} \mid x \prec a\} \\ &\leq a, \end{aligned}$$

which says that $a = a^{**}$. Therefore L is Boolean.

(1) $\Leftrightarrow$ (4): If L is Boolean, $\phi : L \to T$ is a general filter on L, and $a \in L$ such that $\phi(a) = 1$, the $c = a$ is a complemented element of L such that $c \leq a$ and $\phi(c) \neq 0$. Therefore ϕ is Boolean. Conversely, let $0 \neq b \in L$. Consider the general filter $\chi_F : L \to \mathbf{2}$ on L, where F is the classical filter $F = \uparrow b$ in L. By the present hypothesis, χ_F is Boolean. Since $\chi_F(b) = 1$, there exists some complemented $c \leq b$ such that $\chi_F(c) \neq 0$. This in turn implies that $\chi_F(c) = 1$, that is, $c \in \uparrow b$, meaning that $c \geq b$. Consequently, $c = b$ and hence L is Boolean.

(5) $\Leftrightarrow$ (1): If L is Boolean and $\phi : L \to T$ is a prime general filter on L, then for any $a \in L$ we have $a \vee a^* = 1$, which, by primeness, implies that $1 = \phi(a \vee a^*) = \phi(a) \vee \phi(a^*)$, thus showing that ϕ is an ultrafilter. Conversely, suppose, by way of contradiction, that L is not Boolean. Take $a \in L$ such that $a \vee a^* < 1$. By the dual version of the Stone's separation lemma, there is a prime filter $F \subset L$ such that $a \vee a^* \not\in F$. The general filter $\chi_F : L \to \mathbf{2}$ is prime, so, by the present hypothesis, it is an ultrafilter. In consequence, $\chi_F(a) \vee \chi_F(a^*) = 1$, which implies that $a \in F$ or $a^* \in F$, neither of which is possible since $a \vee a^* \not\in F$. Therefore L is Boolean.

$\square$

4.2 A stronger variant of clustering

The notion of a strong variant of clustering was introduced to address the deficiency regarding clustering of classical filters. In any Boolean frame without atoms, the classical filter $\uparrow a$ $(a \neq 0)$ clusters but is not contained in a convergent filter. There is no similar situation for these in topological spaces. Dube and Naidoo [24] defined a classical filter F in a frame L to be strongly clustered in an effort to address the situation. The definition is as follows.

Definition 4.2.1. A filter F in a frame L *strongly clusters* if $\bigvee\{x^* \mid x \in F\} \leq p$ for some $p \in \Sigma L$.

Theorem 4.2.1. *[16] A filter F in a frame L strongly clusters if and only if there is a completely prime filter $P \subseteq L$ such that $P \subseteq secF$.*

Proof. (⇒): Suppose that F strongly clusters. Then $\bigvee\{x^* \mid x \in F\} \leq p$ for some $p \in \Sigma L$. Then $P = \{x \in L \mid x \nleq p\}$ is completely prime. If $x \in P$, $x \wedge y = 0$ for some $y \in F$, then $x \leq y^* \leq \bigvee\{x^* \mid x \in F\} \leq p$ which is a contradiction. Thus for each $x \in P$, $x \wedge y \neq 0$ for each $y \in F$. Thus $x \in secF$ so that $S \subseteq secF$.

(⇐): If $Q \subseteq secF$ for some completely prime filter Q in L, then $p = \bigvee(L \backslash Q) \in \Sigma L$. If $\bigvee\{x^* \mid x \in F\} \in Q$, since Q is completely prime, $x^* \in Q$ for some $x \in F$. Then $x^* \in secF$ which is a contradiction. Thus $\bigvee\{x^* \mid x \in F\} \notin Q$ and hence $\bigvee\{x^* \mid x \in F\} \leq p$. Hence F strongly clusters. ◻

Motivated by the results of Dube and Naidoo in [24], that a classical filter in a regular frame strongly clusters if and only if it is contained in a convergent filter, Bhattacharjee and Naidoo formulated the following definition:

Definition 4.2.2. A general filter $\phi : L \to T$ *strongly clusters (or is strongly clustered)* if there is a convergent general filter $\tau : L \to T$ such that $\phi \leq \tau$.

The relationship of strong clustering to other properties of general filters are given in the proposition below.

Proposition 4.2.1. *[16] The following properties hold.*

1. *If a general filter strongly clusters, then it clusters.*

2. *A maximal general filter (and, hence an ultrafilter) strongly clusters if and only if it converges.*

3. *A prime general filter on a regular frame strongly clusters if and only if it strongly converges.*

Proof. 1. Let $\phi : L \to T$ be a strongly clustered general filter. By definition, there is a convergent $\tau : L \to T$ such that $\phi \leq \tau$. Let C be a cover of L. We cannot have $\phi(c^*) = 1$ for every $c \in C$ as that would imply

$$\begin{aligned} 1 &= \bigwedge_{c\in C} \phi(c^*) \\ &\leq \bigwedge_{c\in C} \tau(c^*) \\ &\leq \bigwedge_{c\in C} \tau(c)^* \\ &= \Big(\bigvee \tau(c)\Big)^* \\ &= 0. \end{aligned}$$

We conclude therefore that ϕ is clustered.

2. This is immediate.

3. Since strong convergence implies convergence, which, in turn, implies strong clustering, only one implication needs to be shown. So suppose that $\phi : L \to T$ is a strong clustering prime filter on a regular frame L. Take a convergent $\tau : L \to T$ such that $\phi \leq \tau$. As shown in Banaschewski and Hong [10], the map $\tau^\circ : L \to T$ defined by $\tau^\circ(a) = \bigvee\{\tau(x) \mid x \prec a\}$ is a frame homomorphism. We show that $\tau^\circ \leq \phi$. Let $b \in L$, and consider any $x \prec b$. Then $x^* \vee b = 1$, which, by the primeness of ϕ, implies $1 = \phi(x^*) \vee \phi(b) \leq \tau(x^*) \vee \phi(b)$, whence $\tau(x) \leq \phi(b)$, consequently $\tau^\circ(b) \leq \phi(b)$, by the regularity of L. Therefore strongly converges.

□

Corollary 4.2.1. *[16] The following are equivalent for a regular frame L:*

1. *L is compact.*
2. *Every prime general filter on L strongly clusters.*
3. *Every general ultrafilter on L strongly clusters.*

Proof. (1) ⇒ (2): Since every cover of a compact frame admits a finite cover, every prime general filter on a compact frame takes covers to covers, and therefore strongly clusters.

(2) ⇒ (3): This follows from the fact that every general ultrafilter is prime (Theorem 4.1.1).

(3) ⇒ (1): If every general ultrafilter on L strongly clusters, then every general ultrafilter on L clusters, and so L is almost compact by Corollary 4.1.1. Since L is regular, this means that L is compact. □

4.3 F-compactness

In this section we consider compactness in terms of convergence of filters. In particular, we consider filters of certain type and require that all filters of that type be convergent. This exercise is briefly known as filter selection.

Definition 4.3.1. [12] An object function F on the category of frames is called a *filter selection* if $\mathrm{F}(L)$ is a class of filters $\phi\colon L \to T$ for each frame L such that:

1. every frame homomorphism $L \to M$ belongs to $\mathrm{F}(L)$;
2. $\mathrm{F}(L)$ is closed under composition, that is, for any $\phi\colon L \to M$ in $\mathrm{F}(L)$ and $\psi\colon M \to N$ in $\mathrm{F}(M)$, $\psi \circ \phi$ belongs to $\mathrm{F}(L)$.

Definition 4.3.2. For any filter selection F of filters, a frame L is called an F$-$*lax retract* of a frame M if there exists a frame homomorphism $h : L \to M$ and a filter $\phi : M \to L$ in F(M) such that $\phi \circ h \leq id_L$.

Definition 4.3.3. [12] For any filter selection F, a frame L is called F-*compact* if every $\phi \in$ F(L) is convergent and we say L is *strongly* F-*compact* if every $\phi \in$ F(L) is strongly convergent.

Proposition 4.3.1. *[12] For any filter selection* F*, closed quotients of* F*-compact frames are* F*-compact.*

Proof. Let L be any F-compact frame, $a \in L$ and $v : L \to \uparrow a$ be defined by $v(x) = x \vee a$ for all $x \in L$. For any $\tau : \uparrow a \to T$ in F$(\uparrow a)$, since F is a filter selection, we have $\tau \circ v \in$ F(L) which implies that $\tau \circ v$ is convergent since L is F-compact. Now, let $C \subseteq \uparrow a$ be a cover of $\uparrow a$, therefore C is also a cover of L and $x \geq a$ for all $x \in C$. Then

$$
\begin{aligned}
v(C) &= \{x \vee a \mid x \in C\} \\
&= a \vee C \\
&= C \text{ since a is the bottom element in } \uparrow a. \text{ Therefore}
\end{aligned}
$$

$$\tau \circ v(C) = \tau\big(v(C)\big) = \tau(C)$$

which implies that $\tau(C)$ is a cover of T (by convergence of $\tau \circ v$) and hence τ is convergent. Hence $\uparrow a$ is F-compact. $\square$

Proposition 4.3.2. *[12] For any filter selection* F*,* F*-lax retracts of* F$-$*compact frames are* F$-$*compact.*

Proof. Let M be an F$-$compact frame, $h : L \to M$ be any frame homomorphism such that $\psi \circ h \leq id_L$ for some $\psi : M \to L$ and $\phi : L \to T$ in F(L). Then $\phi \circ \psi \in$ F(M), making it convergent and hence also $\phi \circ \psi \circ h$ is convergent. $\square$

Proposition 4.3.3. *[12] This also holds for strong* F*-compactness, that is, closed quotients of strongly* F*-compact frames are strongly* F*-compact.*

Proof. Suppose that L is a strongly F-compact frame and consider any $\tau : \uparrow a \to T$ in $\mathrm{F}(\uparrow a)$. Then $\tau \circ v : L \to T$ belongs to $\mathrm{F}(L)$ and since L is strongly F-compact, there exists a frame homomorphism $h : L \to T$ such that $h \leq \tau \circ v$. Then

$$\begin{aligned} h(a) &\leq \tau \circ v(a) \\ &= \tau\big(v(a)\big) \\ &= \tau(a) \text{ since } v(a) = a \vee a = a \\ &= 0_T \text{ since } a = 0_{\uparrow a} \end{aligned}$$

therefore $h(a) \leq 0_T \Longrightarrow h(a) = 0_T$ hence h factors through v, which means $h = k \circ v$ for some $k : \uparrow a \to T$. Hence $k \leq \tau$, showing that $\uparrow a$ is strongly F-compact. $\square$

Proposition 4.3.4. *[12]* F*-lax retracts of strongly* F*-compact frames are strongly* F*-compact.*

Proof. If M is strongly F$-$compact and $h : L \to M$, $\psi : M \to L$ exhibit L as an F$-$lax retract of M, then for any $\phi : L \to T$ in $\mathrm{F}(L)$, again $\phi\psi \in \mathrm{F}(M)$ and hence $k \leq \phi\psi$ for some frame homomorphism $k : M \to T$ by the present hypothesis. It follows that $k \circ h \leq \phi \circ \psi \circ h \leq \phi$, the latter since $\psi \circ h \leq id_L$, showing ϕ is strongly convergent which proves L is strongly F$-$compact. $\square$

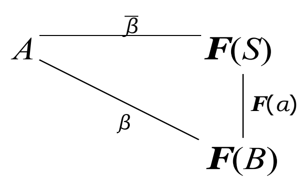
Lemma 4.3.1. *[12] For any frame homomorphism $h : L \to T$ and any filter $\phi : L \to T$, if $h \mid X \leq \phi \mid X$ for some generating set X of L, then $h \leq \phi$.*

Proof. Let $M = \{x \in L \mid h(x) \leq \phi(x)\}$. Then $0, 1 \in M$ and $x \wedge y \in M$ for any $x, y \in M$. Further, for any subset S of M, $h(\bigvee S) = \bigvee\{h(t) \mid t \in S\} \leq \bigvee\{\phi(t) \mid t \in S\} \leq \phi(\bigvee S)$ and hence $\bigvee S \in M$. Thus M is a subframe of L, and since it contains the generating set X of L it is equal to L. $\square$

For the next few results we first need the following definitions.

Definition 4.3.4. In a category C, a *coproduct* of a collection of object A_i, $i \in J$, is a system of morphisms $q_i \colon A_i \to A$, $i \in J$ such that for every system $f_i \colon A_i \to X$, $i \in J$, in C there is exactly one $f \colon A \to X$ such that $fq_i = f_i$ for all $i \in J$.

Definition 4.3.5. Let $\boldsymbol{F} \colon \mathrm{D} \to \mathrm{C}$ be a functor and $A \in Obj(\mathrm{C})$. A subset $\mathsf{S} \subseteq Obj(\mathrm{D})$ is a *solution set* for A with respect to $\boldsymbol{F}$ if for each $B \in Obj(\mathrm{D})$ and for each $\beta \colon A \to \boldsymbol{F}(B)$ in C there are $S \in \mathsf{S}$, $\alpha \colon S \to B$ and $\overline{\beta} \colon A \to \boldsymbol{F}(S)$ such that the diagram



commutes.

is a *Solution set* for A with respect to

Proposition 4.3.5. *[12] For any filter selection* F*, coproduct of strongly* F*−compact frames are strongly* F*−compact.*

Proof. For any strongly F-compact L_α, let $L = \bigoplus L_\alpha$ with coproduct maps $i_\alpha : L_\alpha \to L$ and consider any $\phi : L \to T$ in $\mathrm{F}(L)$. Then $\phi \circ i_\alpha : L_\alpha \to T$ belongs to $\mathrm{F}(L_\alpha)$ so that there exist frame homomorphisms $h_\alpha \colon L_\alpha \to T$ below $\phi \circ i_\alpha$, and choosing such $h_\alpha \colon L_\alpha \to T$ for each α we obtain a frame homomorphism $h : L \to T$ such that $h \circ i_\alpha = h_\alpha$. It then follows that $h \circ i_\alpha \leq \phi \circ i_\alpha$ for each α, and since L is generated by the union of the $Im(i_\alpha)$ and the previous lemma shows that $h \leq \phi$. $\square$

Proposition 4.3.6. *[12] For any filter selection* F*, the* F*-compact regular frames are coreflective in* ***Frm****.*

Proof. Note that F−compactness = strong F−compactness and strong convergence is unique in this setting. Also, since the regular frames are coreflective in **Frm** it is enough to argue within the category **RFrm** of these frames. Now, since coequalizers in **RFrm** are closed quotients the previous two propositions show that the only thing left to check here is the existence of a Solution Set. We claim this is provided, for any frame L, by the set of all F−compact regular quotients of the down set frame $\mathcal{D}L$ of L. To see this, let $h : M \to L$ be any frame homomorphism from an F-compact regular frame M into L and consider its dense-onto factorization

$$h : M \xrightarrow{v} \uparrow s \xrightarrow{k} L$$

where $s = h_*(0)$, $v = (\cdot) \vee s$, and k such that $k \circ v = h$. Then $\uparrow s$ is F−compact regular, as closed quotient of M, and k is dense. It follows that the right adjoint $k_* : L \to \uparrow s$ of k is a filter and hence induces a frame homomorphism $l : \mathcal{D}L \to \uparrow s$ such that $l(\downarrow a) = k_*(a)$. Furthermore, $\uparrow s$ is generated by $Im(k_*)$ since it is regular and consequently l is onto. In all this shows $\uparrow s$ is isomorphic to a quotient of $\mathcal{D}L$ which proves the claim. $\square$

Remark 4.3.1. [12] The same proof leads to the corresponding result for completely regular and for zero-dimensional frames.

For any frame L, F(L) determines a nucleus $n_{\mathrm{F}L}$ defined by

$$n_{\mathrm{F}L}(U) = \bigcap \{\overline{\phi}_*\overline{\phi}(U) \mid \phi \in \mathrm{F}(L)\}$$

where $\overline{\phi} : \mathcal{D}L \to T$ is the frame homomorphism associated with the filter $\phi : L \to T$ and $\overline{\phi}_*$ is its right adjoint so that $\overline{\phi}_*\overline{\phi}$ is the nucleus determined ϕ. Note that $n_{\mathrm{F}L}(U) \subseteq \downarrow(\bigvee U)$ ($\bigvee$ taken in L) because $id_L \in$ F(L) and hence $\bigvee n_{\mathrm{F}L}(U) = \bigvee U$, showing that the homomorphism $\bigvee : \mathcal{D}L \to L$ given by taking joins in L induces a frame homomorphism $k : Fix(n_{\mathrm{F}L}) \to L$. Further, $n_{\mathrm{F}L}(\downarrow a) = \downarrow a$ for each $a \in L$, and consequently we have $\downarrow : L \to Fix(n_{\mathrm{F}L})$. In the following, set $\mathcal{F}L = Fix(n_{\mathrm{F}L})$.

Proposition 4.3.7. *[12] For any natural filter selection* F*, the following are equivalent.*

1. *L is strongly $\mathrm{F}-$compact.*
2. *$\downarrow: L \to FL$ is strongly convergent.*
3. *L is a lax retract of FL.*

Proof. (1) ⇒ (2): Immediate by definition of natural F.
(2) ⇒ (3): By hypothesis, there exists a frame homomorphism $h : L \to FL$ such that $h \leq \downarrow$, and for the homomorphism $k: FL \to L$ introduced above we then have $k \circ h \leq id_L$.
(3) ⇒ (1): By Proposition 4.3.4, it is enough to show that FL is strongly $\mathrm{F}-$compact. Consider then any $\phi : FL \to T$ in $\mathrm{F}(FL)$. Then $\psi = \phi \circ \downarrow: L \to T$ belongs to $\mathrm{F}(L)$ by naturality of F and hence the corresponding homomorphism $\psi^{\mathrm{I}} : DL \to T$ determines a homomorphism $h : FL \to T$ such that $h \circ n_{\mathrm{F}L} = \psi^{\mathrm{I}}$. Now

$$h(\downarrow a) = h \circ n_{\mathrm{F}L}(\downarrow a) = \psi^{\mathrm{I}}(\downarrow a) = \psi(a) = \phi(\downarrow a),$$

and since the $\downarrow a$ generates FL and by Lemma 4.3.1 implies that $h \leq \phi$, showing FL is strongly $\mathrm{F}-$compact. ∎

Proposition 4.3.8. *[12] All these filter selections are natural.*

Proof. We begin with a more general consideration. Let U be any collection of subsets of a frame L such that $\{a \wedge t \mid t \in S\} \in U$ for each $S \in U$ and $a \in L$, and call filter $\phi : L \to T$ U-prime if $\phi(\bigvee S) = \bigvee \phi(S)$ for all $S \in U$. Further, let $C \subseteq DL$ be the closure system of all $U \in DL$ such that $S \subseteq U$ implies $\bigvee S \in U$ for all $S \in U$ and l the corresponding closure operator on DL. Then l is a nucleus and C a frame, as is readily seen by the fact that the operator l_0 on DL such that

$$l_0(U) = U \cup \bigcup\{\downarrow(\bigvee S) \mid S \subseteq U \text{ in } U\}$$

is a prenucleus with $Fix(l_0) = Fix(l) = C$. For this, note that trivially $U \subseteq l_0(U)$ and $l_0(U) \subseteq l_0(W)$ whenever $U \subseteq W$, while $l_0(U) \cap W \subseteq l_0(U \cap W)$ because $a \leq \bigvee S$ for $S \subseteq U$ in $\mathcal{U}$ and $a \in W$ implies that $\{a \wedge t \mid t \in S\} \subseteq U \cap W$ which belongs to $\mathcal{U}$ and has join a.

Further, $\downarrow a \in C$ for each $a \in L$ so that we have the filter $\downarrow : L \to C$, and since

$$\bigvee \{\downarrow t \mid t \in S\} = l(\bigcup \{\downarrow t \mid t \in S\}) = \downarrow(\bigvee S)$$

for any $S \in \mathcal{U}$ (where the first join is in C) this is $\mathcal{U}$-prime. Finally, for any $\mathcal{U}$-prime filter $\phi : L \to T$, the induced frame homomorphism $\phi^{\prime} : \mathcal{D}L \to T$ has the property that $\phi^{\prime}(l_0(U)) = \phi^{\prime}(U)$, as seen by straightforward calculation, and consequently also $\phi^{\prime}(l(U)) = \phi^{\prime}(U)$. It therefore follows that $l(U) \subseteq \phi^{\prime}_* \circ \phi^{\prime}(U)$ which shows that

$$l(U) = \bigcap \{\phi^{\prime}_* \circ \phi^{\prime}(U) \mid \phi : L \to T \ \mathcal{U}-\text{prime filter}\}$$

equality since $\downarrow : L \to C$ is one of the ϕ and the term corresponding to it is actually $l(U)$. Now, if F is any of the above filter selection then, for any frame L, the condition assumed above for $\mathcal{U}$ clearly holds for the $S \subseteq L$ specified in each of these cases. It follows that $\downarrow : L \to \mathcal{F}L$ corresponds to the above $\downarrow : L \to C$ and hence belongs to $\mathrm{F}(L)$, showing F is natural, as claimed. $\square$

Remark 4.3.2. [12] The above proof also identifies the corresponding $\mathcal{F}L$ for the different F involved here as follows:

A $-$ $\mathcal{D}L$

P $-$ $\mathcal{I}L$

S the frame $\mathcal{H}L$ of σ-ideals

D the frame $\mathcal{G}L$ of Scott closed downsets

Recall that a frame L is *supercompact* if each cover of L contains 1_L.

Proposition 4.3.9. *[12] A frame L is:*

1. A-*compact if and only if it is supercompact,*
2. P-*compact if and only if it is compact,*
3. S-*compact if and only if it is Lindelof.*

Proof. 1. ($\Rightarrow$): In particular, $\downarrow : L \to DL$ is convergent so that $\{\downarrow s \mid s \in C\}$ is a cover of DL for any cover C of L but $\bigcup\{\downarrow a \mid s \in C\} = \downarrow 1$ implies $1 \in C$, showing L is supercompact.

($\Leftarrow$): Since any cover C of L contains 1 and the same holds for $\phi(C)$ where $\phi : L \to T$ is any filter so that ϕ is trivially convergent.

2. ($\Rightarrow$). Since $\downarrow : L \to IL$ is convergent, $\{\downarrow s \mid s \in C\}$ is a cover of $I(L)$ for anycover C of L and hence the ideal generated by it is $\downarrow 1$. Consequently, there exists $s_1, s_2, \ldots, s_n \in C$ such that $s_1 \vee s_2 \vee \ldots \vee s_n = 1$, showing L is compact.

($\Leftarrow$): Any bounded lattice homomorphism $\phi : L \to T$ takes any finite cover to a cover, and for compact L this says it takes every cover to a cover, that is, it is convergent.

3. ($\Rightarrow$): Again, since $\downarrow : L \to HL$ is convergent, $\{\downarrow s \mid s \in C\}$ is a cover of HL for any cover C of L and hence the $\sigma-$ideal generated by it is $\downarrow 1$. Further, if the Axiom of Countable Choice is assumed this $\sigma-$ideal consists of all $a \leq \bigvee X$ for countable $X \subseteq C$ and hence C has a countable subcover.

($\Leftarrow$): This is the same as in 2.

$\square$

Recall that an element a in a frame L is *compact* if for any $S \subseteq L$, $a \leq \bigvee S$ implies there exist a finite $T \subseteq S$ such that $a \leq \bigvee T$. We denote by $K(L)$ the set of all compact elements of L. Thus L is called *algebraic* if for all $a \in L$, $a = \bigvee_{i \in I} x_i$ for some $x_i \in K(L)$ and L is called *compact* if its top element is compact.

Definition 4.3.6. A *coherent* frame is a compact algebraic frame L such that $a \wedge b \in K(L)$ for all $a, b \in K(L)$.

Proposition 4.3.10. *[12] A frame L is:*

1. *strongly* A*-compact if and only if it is supercompact,*
2. *strongly* P*-compact if and only if it is a lax retract of a coherent frame, and*
3. *strongly* S*-compact if and only if it is a lax retract of a $\sigma-$coherent frame.*

Proof. 1. It is immediate from the previous proposition together with the fact that A-compactness = strong A-compactness for supercompact frames.

2. ($\Rightarrow$): By Proposition 4.3.7 and Proposition 4.3.8, L is a lax retract of its ideal lattice $\mathcal{I}(L)$ which is coherent.

   ($\Leftarrow$): By Proposition 4.3.4, it is enough to prove that any coherent frame M is strongly P$-$compact. Let $\phi : M \to T$ then be any bounded lattice homomorphism. Then its restriction to the sublattice $K \subseteq M$ of all compact elements induces aframe homomorphism $f : \mathcal{I}K \to T$ such that $f(\downarrow c) = \phi(c)$ for each $c \in K$. Further,if $g : M \to \mathcal{I}K$ is the inverse of the familiar isomorphism $\bigvee : \mathcal{I}K \to M$ and $h = f \circ g : M \to T$ then

$$h(c) = f \circ g(c) = f(\downarrow c) = \phi(c)$$

for all $c \in K$, proving $h \leq \phi$ by Lemma 4.3.1 since K generates M.

3. The proof is the exact analogue of that of 2, with ideal lattices replaced by $\sigma-$ideal lattices and compact elements by Lindelöf elements, using the Axiom of Countable Choice in the appropriate places.

□

Recall that a continuous frame L is *stably continuous* if L is compact and $x \ll a \wedge b$ whenever $x \ll a$ and $x \ll b$.

Corollary 4.3.1. *1. Every stably continuous frame is strongly* P*-compact.*

2. Every stably σ-continuous frame is strongly S*-compact.*

Proposition 4.3.11. *The Prime Ideal Theorem holds iff every frame in which all classical prime filters are convergent is compact.*

Proof. We only need to show ($\Leftarrow$) and do this by proving that the hypothesis implies the Tychonoff Product Theorem for compact Hausdorff spaces, which is known to be equivalent to the Prime Ideal Theorem. For this, let X_a ($\alpha \in I$) be any family of such spaces, $X = X_a$, and $h : \mathrm{O}X_a \rightarrow \mathrm{O}X$ the homomorphism induced by the maps $\mathrm{O}X_a \rightarrow \mathrm{O}X$ resulting from the product projections $X \rightarrow X_a$. Now, for any classical prime filter $\phi : \mathrm{O}X \rightarrow \mathbf{2}$, $\phi \circ h$ strongly converges since $\mathrm{O}X_a$ is compact regular, saying that $\xi \leq \phi \circ h$ for some homomorphism $\xi : \mathrm{O}X_a \rightarrow \mathbf{2}$. On the other hand, h is known to bethe reflection map to spatial frames; hence there exists a homomorphism $\zeta : \mathrm{O}X \rightarrow \mathbf{2}$ suchthat $\xi = \zeta \circ h$, hence h is onto it follows that $\zeta \leq \phi$. Now, the given hypothesis shows that $\mathrm{O}X$, and hence X, is compact, as desired. □

Chapter 5

Ideals in $\mathbb{R}L$ and compact frames

In this chapter we study fixed ideals and strongly fixed ideals in $\mathbb{R}L$. In the ring $C(X)$, the maximal ideals are precisely the fixed ones for a compact Hausdorff space X. On the other hand, if every maximal ideal is fixed, then X is compact. A Tychonoff space X is compact if and only if every proper ideal in $C(X)$ is fixed. Our goal here is to study these in the setting of pointfree topology.

5.1 The cozero map

Definition 5.1.1. The frame of reals, denoted $L(\mathbb{R})$, is the frame generated by all ordered pairs $(p, q) \in \mathbb{Q}^2$ such that

1. $(p, q) \wedge (r, s) = (p \vee r, q \wedge s)$;
2. $(p, q) \vee (r, s) = (p, s)$, whenever $p \leq r < q < s$;
3. $(p, q) = \bigvee\{(r, s) \mid p < r < s < q\}$;
4. $1 = \bigvee\{(r, s) \mid p, q \in \mathbb{Q}\}$;

A continuous real-valued function on L is a frame homomorphism $L(\mathbb{R}) \to L$. The elements of the ring $\mathcal{R}L$ are real-valued continuous functions on L, and the operation on $\mathcal{R}L$ is $\circ \in \{+, \cdot, \wedge, \vee\}$ such that for $\alpha, \beta \in \mathcal{R}L$

$\alpha \circ \beta(p, q) = \bigvee\{\alpha(r, s) \wedge \beta(u, w) \mid (r, s) \circ (u, w) \leq (p, q)\}$, where $(r, s) \circ (u, w) \leq (p, q)$ means that for each $r < x < s$ and $u < y < w$ we have $p < x \circ y < q$. For any $r \in \mathbb{R}$, define the constant map $\mathbf{r} \in \mathcal{R}L$ by $\mathbf{r} = \begin{cases} 1 & \text{for } p < r < q, \\ 0 & \text{otherwise} \end{cases}$

Definition 5.1.2. The *cozero map* is the map $coz : \mathcal{R}L \to L$ given by

$\operatorname{coz}(\alpha) = \bigvee\{\alpha(p, 0) \vee \alpha(0, q) \mid p, q \in \mathbb{Q}\} = \alpha(-, 0) \vee \alpha(0, -)$, where

$(0, -) = \bigvee\{(0, q) \mid q \in \mathbb{Q}, q > 0\}$ and

$(-, 0) = \bigvee\{(p, 0) \mid p \in \mathbb{Q}, p < 0\}$.

A cozero element of a frame L is an element of the form coz (α) for some $\alpha \in \mathcal{R}L$. And the cozero part of a frame L, denoted Coz L, is the sublattice of L consisting of all cozero elements of L.

Lemma 5.1.1. *[19] A frame L is completely regular if and only if Coz L generates L.*

Proof. ($\Rightarrow$): Suppose that L is completely regular and let $a \in L$. Then by complete regularity, $a = \bigvee\{x \in L \mid x \prec\prec a\}$. Thus for each $x \prec\prec a$, there is a $c \in$ Coz L such that $x \prec\prec c \prec\prec a$ by [[5], Proposition 2.1.4] thus $a = \bigvee\{x \in L \mid x \prec\prec a\} = \bigvee\{x \in \text{Coz } L \mid x \prec\prec a\}$ and hence Coz L generates L.

($\Leftarrow$): Suppose that Coz L generates L and let $a \in L$. We assume without loss of generality that $0 \neq a \neq 1$. Then there is an element $x \in L$ such that $x \prec a$. Then $x^* \vee a = 1$. By hypothesis we find $c_1 \prec\prec x^*$ and $c_2 \vee a$ such that $c_1 \vee c_2 = 1$. By [[5] , Corollary 5.1.3] find $b_i \in$ Coz L such that $b_i \prec\prec c_i$ and $b_1 \vee b_2 = 1$. Thus $a = \bigvee\{b_i \in \text{Coz } L \mid b_i \prec\prec a\} = \bigvee\{b_i \in L \mid b_i \prec\prec a\}$ and hence L is completely regular. $\square$

Proposition 5.1.1. *[26] Let L be a frame. Then for any $\alpha, \beta \in \mathsf{R}L$, we have*:

1. *$coz(0) = 0$ and $coz(1) = 1$*
2. *$coz(\alpha\beta) = coz(\alpha) \wedge coz(\beta)$;*
3. *$coz(\alpha + \beta) \leq coz(\alpha) \vee coz(\beta)$;*
4. *$coz(\alpha + \beta) = coz(\alpha) \vee coz(\beta)$, if $\alpha, \beta \geq 0$;*
5. *$coz(|\alpha|) = coz(\alpha)$;*
6. *$coz(\alpha \wedge \beta) = coz(\alpha) \wedge coz(\beta)$, if $\alpha, \beta \geq 0$;*

An element f of $\mathsf{R}L$ is said to be bounded if there exists $n \in \mathbb{N}$ such that $f(-n, n) = 1$. The set of all bounded elements of $\mathsf{R}L$ is denoted by $\mathsf{R}^*(L)$ which is a sub-f ring of $\mathsf{R}L$.

Notation 5.1.1. *Let $a \in L$ and $\alpha \in \mathsf{R}L$. Set $L(a, \alpha) = \{r \in \mathbb{Q} \mid \alpha(-, r) \leq a\}$ and $U(a, \alpha) = \{s \in \mathbb{Q} \mid \alpha(s, -) \leq a\}$. Then for any $a \neq 1$, we have $r \leq s$ for eachr $\in L(a, \alpha)$ and $s \in U(a, \alpha)$.*

Proposition 5.1.2. *[27] Let L be a frame. If $p \in \Sigma L$ and $\alpha \in \mathsf{R}L$, then $U(a, \alpha) = \{s \in \mathbb{Q} \mid \alpha(s, -) \leq a\}$ is a Dedekind cut for a real number which is denoted by $\tilde{p}(\alpha)$.*

Proof. Since p is prime, using $\alpha(-, r) \wedge \alpha(r, -) = 0$ we get $L(p, \alpha) \cup U(p, \alpha) = \mathbb{Q}$. Since $\bigvee_{L(p,\alpha)} \alpha(-, r) \leq p$, $L(p, \alpha)$ Q, and similarly, $U(p, \alpha) \neq \mathbb{Q}$. Obviously $L(p, \alpha)$ is a downset and $U(p, \alpha)$ is an upset. $\square$

Proposition 5.1.3. *[27] If p is a prime element of a frame L, then there exists a unique map $\tilde{p}\colon \mathsf{R}L \to \mathbb{R}$ such that for each $\alpha \in \mathsf{R}L$, $r \in L(p, \alpha)$ and $s \in U(a, \alpha)$ we have*

$r \leq \tilde{p}(\alpha) \leq s$.

Recall that an *f-ring* R is a lattice-ordered ring such that $(a \wedge b)c = (ac) \wedge (bc)$ for all a, $b \in R$ and $c \in R^+ = \{x \in R \mid x \geq 0\}$. Also, recall that a *linear map* between two vector spaces U and V over a field F is a map $T : U \to V$ such that $T(u + v) = T(u) + T(v)$ and $T(\alpha u) = \alpha T(u)$ for all $u, v \in U$ and $\alpha \in F$. The proof of the following proposition is found in [27].

Proposition 5.1.4. *[27] If p is a prime element of a frame L, then $\tilde{p}$: $\mathsf{R}L \to \mathrm{R}$ is an onto f-ring homomorphism. Also, p is a linear map with $\tilde{p}(1) = 1$.*

We say an element a of a frame L is *small* if whenever $c \in \mathrm{Coz}\, L$ and $a \vee c = 1$, then $\uparrow c$ is compact. Given a frame L, we set $\mathsf{R}_S(L) = \{\phi \in \mathsf{R}L \mid \mathrm{coz}\ \alpha$ is small$\}$ and $\mathsf{R}_K(L) = \{\phi \in \mathsf{R}L \mid \uparrow(\mathrm{coz}\ \alpha)^*$ is compact$\}$. It is shown in [19] that $\mathsf{R}_K(L) \subseteq \mathsf{R}_S(L)$ and that they coincide for basically disconnected frames.

Lemma 5.1.2. *[19] A necessary and sufficient condition that $a \in L$ be small is that, for each $c \in \mathrm{Coz}\, L$, $\uparrow c$ be compact whenever $\uparrow(a \vee c)$ is compact. Hence, the join of two small cozero elements is small.*

Proof. Clearly the condition is sufficient. To see necessity, it suffices , by complete regular- ity, to show that every cover of $\uparrow c$ by cozero elements of L has a subcover. So suppose $\{x_\alpha \mid \alpha \in A\} \subseteq \mathrm{Coz}\, L$ with $x_\alpha \geq c$ for each α and $\bigvee x_\alpha = 1$. Then $\{a \vee x_\alpha \mid \alpha \in A\}$ is a cover ofthe compact frame $\uparrow(a \vee c)$. Compactness yields finitely many indices $\alpha_1, \alpha_2, \ldots, \alpha_m$ such that $a \vee x_{\alpha_1} \vee x_{\alpha_2} \vee \ldots \vee x_{\alpha_m} = 1$. Since $x_{\alpha_1} \vee x_{\alpha_2} \vee \ldots \vee x_{\alpha_m} \in \mathrm{Coz}\, L$ and s is small, $\uparrow(x_{\alpha_1} \vee x_{\alpha_2} \vee \ldots \vee x_{\alpha_m})$ is compact. Now, $\{(x_{\alpha_1} \vee x_{\alpha_2} \vee \ldots \vee x_{\alpha_m}) \vee x_\alpha \mid \alpha \in A\}$ is a cover of the compact frame $(x_{\alpha_1} \vee x_{\alpha_2} \vee \ldots \vee x_{\alpha_m})$. There are therefore finitely many indices $\beta_1, \beta_2, \ldots, \beta_k$ such that $x_{\alpha_1} \vee x_{\alpha_2} \vee \ldots \vee x_{\alpha_m} \vee x_{\beta_1} \vee x_{\beta_2} \vee \ldots \vee x_{\beta_k} = 1$, which shows that $\uparrow c$ is compact, as required. Now let c_1, c_2 be small cozero elements, and d be a cozero element with $d \vee (c_1 \vee c_2) = 1$. Then $(d \vee c_1) \vee c_2 = 1$, and hence $\uparrow(d \vee c_1)$ is compact since c_2 is small and $d \vee c_1 \in \mathrm{Coz}\, L$. Thus $\uparrow d$ is compact by the first part, and hence $c_1 \vee c_2$ is small.

We show that $\mathrm{R}_s(L)$ is a proper ideal of $\mathrm{R}L$ if and only if L is not compact.

Proposition 5.1.5. *[19] $\mathrm{R}_s(L)$ is an ideal of $\mathrm{R}L$ which is improper if and only if L is compact.*

Proof. ($\Rightarrow$): Clearly, smaller than small is small. Therefore $\alpha\phi \in \mathrm{R}_s(L)$ whenever $\phi \in \mathrm{R}_s(L)$ and $\alpha \in \mathrm{R}L$. Next, if $\phi_1, \phi_2 \in \mathrm{R}_s(L)$, then coz $\alpha_1 \vee$ coz α_2 is small, and hence coz $(\phi_1 + \phi_2)$ is small, so that $\phi_1 + \phi_2 \in \mathrm{R}_s(L)$. For the latter part, if $\mathrm{R}_s(L)$ is improper, then $\mathbf{1} \in \mathrm{R}_s(L)$, and since $0 \in \mathrm{Coz}\, L$ and $0 \vee \mathrm{coz}\, \mathbf{1} = 1$, it follows that $\uparrow 0 = L$ is compact. ($\Leftarrow$): If L is compact, then every element of L is small and so $\mathbf{1} \in \mathrm{R}_s(L)$. $\square$

In a similar fashion, we show that $\mathrm{R}_K(L)$ is a proper ideal of $\mathrm{R}L$ if and only if L is not compact.

Proposition 5.1.6. *[19] $\mathrm{R}_K(L)$ is an ideal of $\mathrm{R}L$ which is improper if and only if L is compact.*

Proof. From Lemma 2.1.1, if $a, b \in L$ such that $\uparrow a$ and $\uparrow b$ are compact then $\uparrow(a \wedge b)$ is also compact. Now, let $\phi_1, \phi_2 \in \mathrm{R}_K(L)$. Then $\uparrow(\mathrm{coz}\, \phi_1)^*$ and $\uparrow(\mathrm{coz}\, \phi_2)^*$ are compact. Therefore

$$\uparrow(\mathrm{coz}\, \phi_1)^* \wedge \uparrow(\mathrm{coz}\, \phi_2)^* = \uparrow\left((\mathrm{coz}\, \phi_1)^* \wedge (\mathrm{coz}\, \phi_2)^*\right) = \uparrow(\mathrm{coz}\, \phi_1 \vee \mathrm{coz}\, \phi_2)^* \text{ is compact.}$$

But $(\mathrm{coz}\, \phi_1 \vee \mathrm{coz}\, \phi_2)^* \leq \mathrm{coz}\, (\phi_1 + \phi_2)^*$; so $\phi_1 + \phi_2 \in \mathrm{R}_K(L)$. Next, if $\phi \in \mathrm{R}_K(L)$ and $\alpha \in \mathrm{R}L$, then $\uparrow(\mathrm{coz}\, \alpha\phi)^*$ is compact since $(\mathrm{coz}\, \alpha\phi)^* \geq (\mathrm{coz}\, \phi)^*$. Lastly, since $(\mathrm{coz}\, \mathbf{1})^* = L$, it follows immediately that $\mathrm{R}_K(L) = \mathrm{R}L$ if and only if L is compact. $\square$

Proposition 5.1.7. *[19] $\mathrm{R}_s(L) = \bigcap\{M \subseteq \mathrm{R}L \mid M$ is a free maximal ideal$\}$.*

Proof. Let $\phi \in \mathsf{R}_s(L)$ and I be a point of βL with $\bigvee I = 1$. We must show that $r(\mathrm{coz}\ \phi) \leq I$. If not, then $r(\mathrm{coz}\ \phi) \leq \vee I = 1_{\beta L}$, and therefore there is a cozero element c in I such that $c \vee \mathrm{coz}\ \phi = 1$. Thus, $\uparrow c$ is compact since $\phi \in \mathsf{R}_s(L)$. But now the set $I^1 = I \cap \mathrm{Coz}\ L$ isa proper ideal of Coz L such that $c \in I^1$ and $\bigvee I^1 = 1$. This violates the lemma, and hence establishes the inclusion $\subseteq$. On the other hand, let ϕ be in the stated intersection. Suppose, for contradiction, that $\phi \not\in \mathsf{R}_s(L)$. Then there is a cozero element c such that $c \vee \mathrm{coz}\ \phi = 1$ but $\uparrow c$ is not compact. By the lemma, select a proper ideal J of Coz L such that $c \in J$ and $\bigvee J = 1$. Put $Q = \{\alpha \in \mathsf{R}L \mid \mathrm{coz}\ \alpha \in J\}$. Clearly Q is a free proper ideal of $\mathsf{R}L$, and so is contained in some free maximal ideal M. Take $\gamma \in \mathsf{R}L$ such that$c = \mathrm{coz}\ \gamma$. Then M contains both γ and ϕ, and hence the invertible element $\gamma^2 + \phi^2$, which is impossible. Therefore the inverse inclusion also holds. □

Corollary 5.1.1. *[19] The following are equivalent*:

1. $\mathsf{R}_s(L)$ *is a prime ideal.*
2. $\mathsf{R}_s(L)$ *is a free maximal ideal.*
3. *βL is a one-point compactification of L.*

Definition 5.1.3. An ideal Q of $\mathsf{R}(L)$ is a *z-ideal* if whenever $\alpha \in Q$ and $\mathrm{coz}\ \alpha = \mathrm{coz}\ \beta$ for some $\beta \in \mathsf{R}(L)$, then $\beta \in Q$.

Lemma 5.1.3. *[19] A z-ideal Q is prime if and only if whenever $\alpha\beta = 0$, then $\alpha \in Q$ or $\beta \in Q$.*

Proof. The forward implication is trivial. Conversely, let Q be a $z-$ideal with the stated property. Then coz $[Q]$ is an ideal of Coz L such that whenever $a \wedge b = 0$ in Coz L, then$a \in$ coz $[Q]$ or $b \in$ coz $[Q]$. Thus, by Lemma 3.8 in [22], coz $[Q]$ is a prime ideal of Coz L. Now, suppose $\alpha\beta \in Q$.

Then $\text{coz}\,\alpha \wedge \text{coz}\,\beta \in \text{coz}\,[Q]$. We may assume that $\text{coz}\,\alpha \in \text{coz}\,[Q]$. Since Q is a $z-$ideal it follows that $\alpha \in Q$. Therefore Q is a prime ideal. □

Proposition 5.1.8. *[19] $\mathsf{R}_K(L)$ is prime if and only if L is a noncompact continuous frame, and whenever $a \wedge b = 0$ in Coz L, then $\uparrow a^*$ or $\uparrow b^*$ is compact.*

Recall that a frame L is *nowhere compact* if, for any $a \in L$, $\uparrow a$ compact implies a is dense.

Proposition 5.1.9. *[19] The following are equivalent for a frame L:*

1. *L is nowhere compact.*
2. *$\mathsf{R}_\infty(L)$ is the zero ideal.*
3. *$\mathsf{R}_K(L)$ is the zero ideal.*
4. *$\mathsf{R}_s(L)$ is the zero ideal.*
5. *For every nonzero $J \in \beta L$, there is a point I of βL with $\bigvee I = 1$ and $I \vee J = 1_{\beta L}$.*

Let R be a commutative ring with unit; then an ideal of R is *pure* if for every $x \in I$ there exists $y \in I$ such that $x = xy$. The operator m is defined on the lattice of ideals of R by $mI = \{a \in R \mid a = ab \text{ for some } b \in I\}$. In general, the ideal mI need not be pure, but in the case of $\mathsf{R}L$, mQ is pure for every ideal Q.

Given an ideal Q of $\mathsf{R}L$, let a_Q denote the element of L defined by $a_Q = \bigvee\{\text{coz}\,\alpha \mid \alpha \in Q\}$. If we write $a_L = \bigvee \text{coz}\,\gamma_n$ with $\text{coz}\,\gamma_n \prec\prec \text{coz}\,\gamma_{n+1}$ for each n. Then we have the following proposition.

Proposition 5.1.10. *[19] $m\mathsf{R}_K(L)$ is finitely generated if and only if a_L is compact.*

Proof. (⇒): Suppose $m\mathsf{R}_K(L) = \langle \phi_k \mid k \in \Lambda, \text{ for some finite } \Lambda \rangle$, then $a_L = \bigvee \text{coz}\,\phi_k$. Fix k. Since $\phi_k \in m\mathsf{R}_K(L)$, then there exists $\phi \in \mathsf{R}_K(L)$ such that $\text{coz}\,\phi_k \prec\prec \text{coz}\,\phi \leq a_L$. Since $\phi_k \in \mathsf{R}_K(L)$, then $\uparrow(\text{coz}\,\phi_k)^*$ is compact. Consequently $\text{coz}\,\phi_k \prec\prec a_L$. Now suppose $T \subseteq L$ such that $a = \bigvee T$, then since $\text{coz}\,\phi_k \prec\prec a_L$, there exists a finite $T_k \subseteq T$ such that $\text{coz}\,\phi_k \leq \bigvee T_k$. Now put $S = \bigcup_{k\in\Lambda} T_k$, then S is a finite subset of T such that $a_L = \bigvee S$, and hence a is compact.

(⇐): If a_L is compact, then there are finitely many elements $\alpha_1, \alpha_2, ..., \alpha_n$ in $m\mathsf{R}_K(L)$ such that $a_L = \text{coz}\,\alpha_1 \vee \text{coz}\,\alpha_2 \vee \cdots \vee \text{coz}\,\alpha_n = \text{coz}\,(\alpha_1^2 + \alpha_2^2 + \cdots + \alpha_n^2)$. We show that $m\mathsf{R}_K(L) = \langle \alpha_1, \alpha_2, \ldots, \alpha_n \rangle$. The one inclusion is trivial. Let $\phi \in m\mathsf{R}_K(L)$, then there exists $\gamma \in \mathsf{R}_K(L)$ such that $\text{coz}\,\phi \prec\prec \text{coz}\,\gamma$. Now take any $\tau \in \mathsf{R}L$ such that $\text{coz}\,\phi \prec\prec \text{coz}\,\tau \prec\prec \text{coz}\,\gamma \leq a_L$, we get

$$\text{coz}\,\phi \prec\prec \text{coz}\,\tau \prec\prec \text{coz}\,(\alpha_1^2 + \alpha_2^2 + \cdots + \alpha_n^2),$$

showing that ϕ is a multiple of $\alpha_1^2 + \alpha_2^2 + \cdots + \alpha_n^2$, establishing the reverse inclusion. $\square$

5.2 Weakly spatial frames

Definition 5.2.1. A frame L is *weakly spatial* if for $a \in L$, $a < 1$ implies that $\Sigma_a \neq \Sigma_1$.

Lemma 5.2.1. *[26] A frame L is weakly spatial if and only if there is a prime element $p \in L$ such that $a \leq p < 1$ for every $a < 1$.*

Proof. Suppose that L is weakly spatial and $a < 1$ for some $a \in L$. Then $\Sigma_a \neq \Sigma_1 = \Sigma L$, which implies that there is a prime element $p \in \Sigma L$ such that $p \not\in \Sigma_a$ and hence $a \leq p$. Conversely, let $a < 1$ and $p \in \Sigma L$ such that $a \leq p$. Then $p \in \Sigma L \backslash \Sigma_a$ and hence $\Sigma L \neq \Sigma_a$ which implies that L is weakly spatial. $\square$

Observation 5.2.1. *[26] We observe that if L is spatial, then it is weakly spatial.*

The following example is due to Estaji [26].

Example 5.2.1. [26] Let L be a nonspatial frame and $M = L \cup \{1_M\}$, where the order of M is the same as in L for the elements of L and for every $x \in L, x < 1_M$. Thetop element 1_L is a prime element of M, so M is weakly spatial for all L. Now since$\Sigma M = \Sigma L \cup \{1_L\}$, M is nonspatial.

Proposition 5.2.1. *[26] Every compact frame is weakly spatial.*

Proof. Let L be a compact frame and $a \in L$ such that $a < 1$. Since every ideal I can be written in the form $I = \{\downarrow x \mid x \in I\}$, by the Axiom of Choice we can find a maximal ideal $P \subset L$ such that $a \in P$. Let $p = \bigvee P$, then by compactness of $L, p < 1$. Since P is maximal, then $\downarrow p = \{x \in L \mid x \leq p\} = P$. Also, P is a prime ideal and hence p is also prime and hence $a \leq p$ implies $\Sigma_a \neq \Sigma L$. $\square$

Lemma 5.2.2. *[26] Let L be weakly spatial and $\alpha \in \mathrm{R}L$. If $\Sigma_{coz(\alpha)=\emptyset}$, then coz $(\alpha) = 0$.*

Proof. Let $r, s \in \mathbb{Q}$ such that $r < 0 < s$ and $p \in \Sigma L$. Suppose $\Sigma_{\mathrm{coz}(\alpha)} = \emptyset$, then $p \notin \Sigma_{\mathrm{coz}(\alpha)}$ which implies that coz $(\alpha) \leq p$. Now, if $\alpha(r, s) \leq p$, then

$$\begin{aligned}
\mathrm{coz}(\alpha) \vee \alpha(r, s) &= \left[\alpha(0, -) \vee \alpha(-, 0)\right] \vee \alpha(r, s) \\
&= \alpha(0, -) \vee \alpha(-, 0) \vee \alpha(r, s) \\
&= \alpha\left((0, -) \vee (-, 0) \vee (r, s)\right) \\
&= \alpha(1) \\
&= 1.
\end{aligned}$$

Which implies that coz $(\alpha) \vee \alpha(r, s) = 1 \leq p$ implying $p = 1$, and this contradicts the fact that $p < 1$. Hence we must have $\alpha(r, s) \nleq p$, which implies that $p \in \Sigma_{\alpha(r,s)}$ implies $\Sigma L \subseteq \Sigma_{\alpha(r,s)}$ and hence $\Sigma L = \Sigma_{\alpha(r,s)}$. Since L is weakly spatial, we conclude that $\alpha(r, s) = 1$.

On the other hand,

$$
\begin{aligned}
0 &= \big(\alpha(-, r) \vee \alpha(s, -)\big) \wedge \alpha(r, s) \\
&= \big(\alpha(-, r) \vee \alpha(s, -)\big) \wedge 1 \\
&= \alpha(-, r) \vee \alpha(s, -).
\end{aligned}
$$

Therefore, $\operatorname{coz}(\alpha) = \bigvee \{\alpha(-, r) \vee \alpha(s, -) \mid r < 0 < s\} = 0.$ $\square$

Corollary 5.2.1. *[26] In a compact frame L, for any $\alpha \in \mathrm{R}L$, if $\Sigma_{\operatorname{coz}(\alpha)=\emptyset}$, then $\operatorname{coz}(\alpha) = 0$.*

Proof. This follows from the fact that every compact frame is weakly spatial. $\square$

Definition 5.2.2. A frame L is *conjunctive* if for every $a \in L$ with $a \nleq b$, there exist $c \in L$ such that $a \vee c = 1$ and $b \vee c \neq 1$.

For background information about conjunctive frames and separation Axioms, see [[33], [37], [35]], although the terminology is different, they call a sublift frame for what we call a conjunctive frame.

Lemma 5.2.3. *[26] A frame L is spatial if and only if for each $a, b \in L$ with $a \nleq b$, there exist a prime element $p \in L$ such that $a \nleq p$ and $b \leq p$.*

Proof. ($\Rightarrow$) Suppose that L is a spatial frame and $a, b \in L$ with $a \nleq b$. Since a spatial frame is weakly spatial, then for every $a < 1$, there is a prime element $p \in L$ such that $a \leq p < 1$. Now $a > b$, so $b < 1$ and $b \leq p$. If a is a point of L, then a is a maximal element below the top, so $a \nleq p$ and $b \leq p$.
($\Leftarrow$): This follows immediately from the definition. $\square$

Proposition 5.2.2. *[26] Let L be a conjunctive frame, then L is a spatial if and only if it is weakly spatial.*

Proof. (⇒): This is immediate (see Observation 5.2.1).

(⇐): Let $a, b \in L$ such that $a \nleq b$. Then there exists $c \in L$ such that $a \vee c = 1$, $b \vee c \neq 1$. Since L is weakly spatial frame, we conclude by Lemma 5.2.1 that there exists a prime element $p \in L$ such that $c \vee b \leq p$. If $a \leq p$, then $c \vee a = 1 \leq p$, which is a contradiction. Hence $a \nleq p$ and $b \leq p$, which follows that L is spatial. ∎

By [[34], Proposition 1.5] every regular frame is conjunctive, so we have the following.

Corollary 5.2.2. *[26] For a regular frame L, the notion of spatiality and weak spatiality coincide.*

Lemma 5.2.4. *[19] Let $c \in CozL$. Then $\uparrow c$ is compact if and only if for any proper ideal I of $CozL$ with $\bigvee I = 1$, $c \notin I$.*

Proof. (⇐): If I is a proper ideal of $CozL$ with $\bigvee I = 1$, then $\{c \vee x \mid x \in I\}$ is a cover of the compact frame $\uparrow c$, and so $c \vee y = 1$ for some $y \in I$. Since I is a proper ideal, it follows that $c \notin I$.

(⇒): Let A be a cover of $\uparrow c$, and put $J = \{u \in CozL \mid u \leq a$ for some $a \in A\}$. Then $\bigvee J = 1$ by complete regularity. Put $I = \{v \in CozL \mid v \leq \bigvee S$ for some finite $S \subseteq J\}$. Then I is an ideal of $CozL$ containing c and such that $\bigvee S = 1$. The current hypothesis therefore implies that $1 \in I$, that is, $\bigvee S = 1$ for some finite $S \subseteq I$. Hence $\bigvee T = 1$ for some finite $T \subseteq A$. ∎

5.3 Maximal, fixed and strongly fixed ideals of $\mathcal{R}L$

Definition 5.3.1. An ideal I of $\mathcal{R}L$ is called *fixed* if $\bigvee_{\alpha \in I} \operatorname{coz}(\alpha) < 1$.

Lemma 5.3.1. *[25] The following are equivalent for a completely regular frame L:*

1. *L is compact.*

2. *Every ideal of* $\mathsf{R}L$ *is fixed.*

3. *Every ideal of* $\mathsf{R}^*(L)$ *is fixed.*

4. *Every maximal ideal of* $\mathsf{R}L$ *is fixed.*

5. *Every maximal ideal of* $\mathsf{R}^*(L)$ *is fixed.*

Proof. (1) $\Rightarrow$ (2): Because 0 is a cozero element which belongs to every ideal of Coz L, and since $\uparrow 0 = L$, a lattice reflection on Lemma 4.5 of [19] shows that 1 and 2 are equivalent.

(2) $\Leftrightarrow$ (3): If (2) holds, then L is compact, and hence $\mathsf{R}L = \mathsf{R}^*(L)$, so that (3) holdsas well. Conversely, let Q be a free ideal in $\mathsf{R}L$. For any $\phi \in Q$, $\phi^2(1+\phi^2)^{-1}$ is an element of $Q \cap \mathsf{R}^*(L)$ with the same cozero element at ϕ. Hence $Q \cap \mathsf{R}^*(L)$ is a free ideal. Consequently, (3) implies (2).

(2) $\Rightarrow$ (4) and (3) $\Rightarrow$ (5): These equivalences follow from the fact that every free ideal is contained in a free maximal ideal.

$\square$

Definition 5.3.2. Let L be a frame and $\alpha \in \mathsf{R}L$. A *zero-set* in L is defined by

$$Z(\alpha) = \{p \in \Sigma L \mid \alpha[p] = 0\}.$$

The collection of zero-sets in L will be denoted by $Z[L]$.

Lemma 5.3.2. *[26] Let p be a prime element of L. For $\alpha \in \mathsf{R}L$, $\alpha[p] = 0$ if and only if coz $(\alpha) \le p$.*

Proof. ($\Rightarrow$): Suppose that $\alpha(p) \;/\!= 0$. If $\alpha[p] > 0$, then there exist a rational number$r$ such that $\alpha[p] \ge r > 0$. Thus, by Proposition 5.1.2, $r \in L(p, r)$, and by definition of $L(p, r)$, $\alpha(-, r) \le p$. Now, if coz $(\alpha) \le p$, we have $1 = \alpha(0, -) \vee \alpha(-, r) \le$ coz $(\alpha) \vee p \le$

$p \vee p = p$ and obtain a contradiction. Therefore coz $(\alpha) \not\leq p$. In the case $\alpha[p] < 0$, the case is similar.

($\Leftarrow$): Suppose that $\alpha[p] = 0$. So by Proposition 5.1.2 for every two rational numbers $r < 0 < s$, we have $r \in L(\alpha, p)$ and $s \in U(\alpha, p)$, and hence $\alpha(-, r) \vee \alpha(s, -) \leq p$. Thus, coz $(\alpha) = \bigvee\{\alpha(-, r) \vee \alpha(s, -)\} \leq p$. $\square$

Definition 5.3.3. Let I be any ideal in $\mathrm{R}(L)$ or $\mathrm{R}^*(L)$. If $\bigcap Z[I] \neq \emptyset$, we call I a *strongly fixed ideal*, and if $\bigcap Z[I] = \emptyset$ then I is a *strongly free ideal*.

Lemma 5.3.3. *[26] If $\Sigma L \neq \emptyset$, then the zero ideal in $\mathrm{R}L$ or $\mathrm{R}^*(L)$ is strongly fixed.*

Proof. If $\Sigma L \neq \emptyset$, then there is a prime element $p \in L$ such that $Z(\alpha) = \{p \in L \mid \alpha[p] = 0\} = \emptyset$ and since $Z(\alpha) \cap Z(\beta) = Z(|\alpha| + |\beta|) = Z(\alpha^2 + \beta^2)$, it follows that $\bigcap Z[I] \neq \emptyset$ if I is a zero ideal, and hence I is strongly fixed. $\square$

We omit the proofs of the following lemmas because they are immediate.

Lemma 5.3.4. *[26] If $Z(\alpha) \neq \emptyset$, then the principal ideal in (α) is strongly fixed.*

Lemma 5.3.5. *[26] If L is a weakly spatial frame, then every strongly free ideal in $\mathrm{R}L$ or $\mathrm{R}^*(L)$ contains nonzero strongly fixed ideals. In fact, if I contains a nonzero function β whose zero set is nonempty, then I contains the nonzero strongly fixed ideals (β).*

Lemma 5.3.6. *[26] No strongly fixed ideal can contain a strongly free ideal. Also, if*

$$\emptyset \neq S \subseteq \Sigma L, \text{ then } \{\alpha \mid S \subseteq Z[\alpha]\} \text{ is strongly fixed.}$$

Proposition 5.3.1. *[26] Every strongly fixed ideal in $\mathrm{R}L$ or $\mathrm{R}^*(L)$ is fixed.*

Proof. Let I be a strongly fixed ideal in $\mathrm{R}L$ or $\mathrm{R}^*(L)$, then $\bigcap Z[I] \neq \emptyset$. Let $p \in \bigcap Z[I]$, then by Lemma 5.3.2, $\bigvee_{\alpha \in I} \mathrm{coz}(\alpha) \leq p < 1$, and hence I is a fixed. $\square$

Example 5.3.1. [26]

1. Let L be a completely regular frame such that $\Sigma L = \emptyset$. Then, every ideal in $\mathsf{R}L$ or $\mathsf{R}^*(L)$ is strongly free.

2. If $\alpha \in \mathsf{R}L$ such that coz $(\alpha) < 1$ and the ideal I of $\mathsf{R}L$ is generated by α, then $\bigvee_{\beta \in I}$ coz $(\beta) \leq$ coz $(\alpha) < 1$, and so I is a fixed ideal in $\mathsf{R}L$.

Proposition 5.3.2. *[26] If L is a weakly spatial frame, then every fixed ideal in $\mathsf{R}L$ or $\mathsf{R}^*(L)$ is a strongly fixed ideal in $\mathsf{R}L$ or $\mathsf{R}^*(L)$.*

Proof. Let I be a fixed ideal in $\mathsf{R}L$. Since L is a weakly spatial frame and $\bigvee_{\alpha \in I}$ coz $(\alpha) < 1$, we conclude by Lemma 5.2.1 that there exists $p \in \Sigma L$ such that $\bigvee_{\alpha \in I}$ coz $(\alpha) \leq p < 1$. Then by Lemma 5.3.2 , $p \in \bigcap Z[I]$, that is, I is a fixed ideal in $\mathsf{R}L$. $\square$

Define $M_p = \{f \in \mathsf{R}L \mid f[p] = 0\}$ for every prime element $p \in L$. In the following proposition, we show that the strongly fixed maximal ideals are exactly the ideals M_p. We regard the Stone-Čech compactification of L, denoted by βL, as the frame of completely regular ideals of L. We denote the right adjoint of the join map $j_L : \beta L \to L$ by r_L and recall that $r_L(a) = \{x \in L \mid x \prec\prec a\}$. We define $M^I = \{\alpha \in C(L) \mid r_L(\text{coz}\,(\alpha)) \subseteq I\}$, for all $1_{\beta L} \neq I \in \beta L$. If $M^I = M^J$, then $I = J$.

Proposition 5.3.3. *[26] Let L be a completely regular frame.*

1. *The strongly fixed maximal ideals of $\mathsf{R}L$ are precisely the ideals M_p, for $p \in \Sigma L$. The ideals M_p are distinct for distinct $p \in \Sigma L$. For each $p \in \Sigma L$, $\mathsf{R}L/M_p$ is isomorphic with the real field $\mathbb{R}$; in fact, the mapping $\alpha + M_p \to \alpha[p]$ is the unique isomorphism of $\mathsf{R}L/M_p$ onto $\mathbb{R}$.*

2. *The strongly fixed maximal ideals of $\mathsf{R}^*(L)$ are precisely the ideals $M_p^* = \{\alpha \in \mathsf{R}^*(L) \mid \alpha[p] = 0\}$ $(p \in \Sigma L)$. The ideals M_p^* are distinct for distinct $p \in \Sigma L$.*

For each $p \in \Sigma L$, $\mathsf{R}^*(L)/M_p^*$ is isomorphic with the real field $\mathbb{R}$; in fact, the mapping $\alpha + M_p^* \to \alpha[p]$ is the unique isomorphism of $\mathsf{R}^*(L)/M_p^*$ onto $\mathbb{R}$.

Proof. M_p is the kernel of the homomorphism $p^{\mathrm{l}} : \mathsf{R}L \to \mathbb{R}$. Since by Proposition 5.1.4,the homomorphism p^{l} is onto the field $\mathbb{R}$, $\mathsf{R}L/M_p \cong \mathbb{R}$. Hence its kernel M_p is a maximal ideal. It is clear that M_p is a strongly fixed ideal for every prime $p \in L$. Therefore, M_p is a strongly fixed maximal ideal. On the other hand, if M is any strongly fixed maximal ideal in $\mathsf{R}L$, then there exists a point $p \in \bigcap Z[M]$. Evidently, $M \subseteq M_p$, which has just been shown to be an ideal. Hence since M is maximal, $M = M_p$. Now, suppose that $p, q \in \Sigma L$ and $M_p = M_q$. So, $M^{r_L(p)} = M_p = M_q = M^{r_L(q)}$, that is $r_L(p) = r_L(q)$. Therefore, we conclude that $p = q$. Thus the ideals M_p are distinct for distinct $p \in \Sigma L$. □

Corollary 5.3.1. *[26] If L is a completely regular frame and M is a maximal ideal in $\mathsf{R}L$, then M is a fixed maximal ideal in $\mathsf{R}L$ if and only if M is a strongly fixed maximal ideal in $\mathsf{R}L$.*

Proof. As in Proposition 3.3 in [21], we have that the fixed maximal ideals in $\mathsf{R}L$ are precisely the ideals M_p for prime elements $p \in \Sigma L$. Now, by Proposition 5.3.3, the proof is complete. □

Lemma 5.3.7. *[26] Every strongly fixed ideal of $\mathsf{R}L$ is contained in a strongly fixed maximal ideal.*

Proof. Every ideal is contained in the maximal ideal, so any strongly fixed ideal in $\mathsf{R}L$ is contained in a strongly fixed maximal ideal. □

Proposition 5.3.4. *[26] Let L be a completely regular frame. The following statements are equivalent:*

1. *L is a spatial frame.*

2. *For every ideal $I \in \mathrm{R}L$, I is a fixed ideal of $\mathrm{R}L$ if and only if I is a strongly fixed ideal of $\mathrm{R}L$.*

3. *Every fixed ideal of $\mathrm{R}L$ is contained in a fixed maximal ideal.*

Proof. (1) ⇒ (3): See Corollary 3.5 in [21].

(1) ⇒ (2): This follows from the fact that a spatial frame is weakly spatial together with Proposition 5.3.2.

(2) ⇒ (1): Let $1 \neq a \in L$. Since L is a completely regular frame, we conclude that there exists $\{\alpha_j\}_{j\in J} \subseteq \mathrm{R}L$ such that $a = \bigvee_{j\in J} \operatorname{coz}(\alpha)$. Put $I = \langle \alpha_j \mid j \in J \rangle$. Then $\bigvee_{\alpha\in I} \operatorname{coz}(\alpha) = a < 1$, that is, I is a fixed ideal of $\mathrm{R}L$. By hypothesis, I is a strongly fixed ideal of $\mathrm{R}L$, and so there exists $p \in \Sigma L$ such that $p \in \bigcap Z[I]$. Thus by Lemma 5.3.2, $a = \bigvee_{\alpha\in I} \operatorname{coz}(\alpha) \leq p < 1$. Therefore, by Lemma 5.2.1, L is a weakly spatial frame. Now, by Corollary 5.2.2, the proof is complete. ∎

Proposition 5.3.5. *[26] Let L be a weakly spatial frame. Then L is a compact frame if and only if ΣL is a compact space.*

Proof. (⇐): Suppose that L is a compact frame, and $\bigcup_{j\in J} \Sigma_{a_j} = \Sigma L$. So $\Sigma_{\bigvee a_j} = \Sigma_1$ since L is weakly spatial, $\bigvee a_j = 1$. Hence, by compactness of L, there exists $j_1, j_2, \ldots, j_n \in J$ such that $a_{j_1} \vee a_{j_2} \vee, \ldots, \vee a_{j_n} = 1$, and so $\Sigma_{a_{j_1}} \cup \Sigma_{a_{j_2}}, \ldots, \cup \Sigma_{a_{j_n}} = \Sigma_1$.

(⇒): Suppose that ΣL is a compact space and $\bigvee a_j = 1$. Hence, $\bigcup \Sigma_{a_j} = \Sigma_{\bigvee a_j} = \Sigma_1 = \Sigma L$. Thus, by compactness of ΣL, there exists $j_1, j_2, \ldots, j_n \in J$ such that $\Sigma_{a_{j_1}} \cup \Sigma_{a_{j_2}}, \ldots, \cup \Sigma_{a_{j_n}} = \Sigma_1$. So $\Sigma_{a_{j_1} \vee a_{j_2} \vee, \ldots, \vee a_{j_n}} = \Sigma_1$. Hence, since L is weakly spatial, $a_{j_1} \vee a_{j_2} \vee, \ldots, \vee a_{j_n} = 1$. Therefore L is compact. ∎

Proposition 5.3.6. *[26] If L is compact and M is a maximal ideal of $\mathrm{R}L$, then there exists a prime element $p \in L$ such that $M = M_p$.*

Proof. Assume that for every prime element p, $M \nsubseteq M_p$. We have that for every $p \in L$ there exists $f_p \in M_p$. So, by Lemma 5.3.2, coz $(f_p) \not\leq p$, and hence $p \in \Sigma_{\text{coz}\,(f_p)}$. Therefore, $\Sigma_{\bigvee_p \text{coz}\,(f_p)} = \bigcup_p \Sigma_{\text{coz}\,(f_p)} = \Sigma L = \Sigma_1$. Hence by weak spatiality, $\bigvee_p \text{coz}\,(f_p) = 1$. So, since L is compact, there are $p_1, p_2, \ldots, p_n \in \Sigma L$ such that coz $(f_{p_1}) \vee \text{coz}\,(f_{p_2}) \vee, \ldots, \vee \text{coz}\,(f_{p_n}) = 1$. Thus by the property of *cozero map*, coz $(f_p^2 + f_p^2 + \ldots + f_p^2) = 1$, and hence $h = f_p^2 + f_p^2 + \ldots + f_p^2 \in M$ is invertible, which is a contradiction. Therefore, $M \subseteq M_p$ for some $p \in \Sigma L$. Since M is maximal, we conclude that $M = M_p$. □

Proposition 5.3.7. *[26] If L is compact and M is a maximal ideal in $\mathrm{R}^*(L)$, then there exists a prime element $p \in L$ such that $M = M_p^*$.*

Proof. It is similar to Proposition 5.3.6. □

There is a homeomorphism $\tau : \Sigma L(\mathrm{R}) \to \mathrm{R}$ such that $r < \tau(p) < s$ if and only if $(r, s) \not\leq p$ for all prime elements p of R and all $r, s \in \mathbb{Q}$ (see Proposition 1 of [[8], page 12]).

Lemma 5.3.8. *[26] Every prime (maximal) element of R is of the form $p_x = \bigvee\{(-, r) \vee (s, -) \mid r, s \in \mathbb{Q}, r \leq x \leq s\}$ for some $x \in \mathrm{R}$, and $\tau(p_x) = x$. In particular, for every $r \in \mathbb{Q}$, $p_r = (-, r) \vee (r, -)$ and $\tau\big((-, \mathrm{r}) \vee (\mathrm{r}, -)\big) = \mathrm{r}$.*

Proof. Since R is a completely regular frame, the prime elements are precisely the maximal elements, and maximal elements are of the form p_x for some $x \in \mathrm{R}$. □

Theorem 5.3.1. *[26] Let L be a weakly spatial frame. Then the following statements are equivalent:*

1. *L is a compact frame.*
2. *Every proper ideal in $\mathrm{R}L$ is strongly fixed.*
3. *Every maximal ideal in $\mathrm{R}L$ is strongly fixed.*

4. *Every proper ideal in* $\mathsf{R}^*(L)$ *is strongly fixed.*

5. *Every maximal ideal in* $\mathsf{R}^*(L)$ *is strongly fixed.*

Proof. (1) ⇒ (2): Let I be a proper ideal in $\mathsf{R}L$. By Proposition 5.3.6, there exists a prime element $p \in L$ such that $I \subseteq M_p$. So, $p \in \bigcap Z[M_p] \subseteq \bigcap Z[I]$. It follows that I is a strongly fixed ideal.

(1) ⇒ (4): is similar to (1) ⇒ (2).

(2) ⇒ (3) and (4) ⇒ (5) are trivial.

First we show that ΣL is a compact space to prove (3) ⇒ (1). For this, we prove that every maximal ideal M in $\mathsf{R}(\Sigma L)$ is of the form M_x for some $x \in \Sigma L$. Define $\phi : \mathsf{R}L \to \mathsf{R}(\Sigma L)$ by $\phi(f) = \tau \circ \Sigma f = \tau \circ f_*$, where $\tau : \Sigma \mathcal{R} \to \mathbb{R}$ is the homomorphism discussed in Lemma 5.3.8 and $f_* : L \to \mathcal{R}$ is the right adjoint of f. By hypothesis, there is a prime element $p \in L$ such that $\phi^{-1}(M) \subseteq M_p$, so $M \subseteq \phi(M_p)$. Hence $\bigcap\{Z(f) \mid f \in \phi(M_p)\} \subset \bigcap\{Z(f) \mid f \in M\}$. Now, it is enough to show that $\bigcap\{Z(f) \mid f \in \phi(M_p)\} = \emptyset$. Let $f \in M_p$, then $f[p] = 0$ and by Lemma 5.3.2 coz $(\alpha) \leq p$, that is to say, $f((0,-) \vee (-,0)) \leq p$. So $(0,-) \vee (-,0) \leq f_*(p)$, thus since $(0,-) \vee (-,0)$ is a maximal element of $\mathcal{R}$ and $f_*(p)$ is a prime element, $(0,-) \vee (-,0) = f_*(p)$. Now, by Lemma 5.3.8, we have $0 = \tau((0,-) \vee (-,0)) = \tau f_*(p) = \phi(f)$. Therefore $p \in \bigcap\{Z(f) \mid f \in \phi(M_p)\} \subset \bigcap\{Z(f) \mid f \in M\}$. So $M = M_x$. Hence every maximal ideal of $C(\Sigma L)$ is fixed, thus ΣL is compact. Since L is weakly spatial, by Proposition 5.3.5, L is compact. (5) ⇒ (1) is similar to (3) ⇒ (1). $\square$

Remark 5.3.1. Let $\mathsf{M}(\mathsf{R}L)$ denote the set of all maximal ideals in $\mathsf{R}L$. We make $\mathsf{M}(\mathsf{R}L)$ into a topological space by taking, as a base for the closed sets, all sets of the form

$$F(\alpha) = \{M \in \mathsf{M}(\mathsf{R}L) \mid \alpha \in M\} \ (\alpha \in \mathsf{R}L).$$

Define $\Theta\colon \Sigma L \to \mathsf{M}(\mathsf{R}L)$ by $\Theta(p) = M_p$. If L is a compact completely regular frame, then by Proposition 5.3.3 and Theorem 5.3.1, Θ is one-one and onto, respectively. Also, $\Theta^{-1}(\mathsf{F}(\alpha)) = Z(\alpha)$ and $\Theta(Z(\alpha)) = \mathsf{F}(\alpha)$. Therefore, ΣL and $\mathsf{M}(\mathsf{R}L)$ are isomorphic.

Proposition 5.3.8. *[26] Suppose that L and L^{I} are two compact completely regular frames. Then the following statements are equivalent:*

1. $L \cong L^{I}$.

2. *ΣL and ΣL^{I} are homeomorphic.*

3. *$\mathsf{R}L$ and $\mathsf{R}(L^{I})$ are isomorphic.*

Proof. (1) $\Leftarrow\Rightarrow$ (2): Since every compact completely regular frame is spatial, we conclude that $L \cong O\Sigma L$ and $L^{I} \cong O\Sigma L^{I}$.

(1) $\Rightarrow$ (3): Is immediate.

(3) $\Rightarrow$ (2): Let $\phi\colon \mathsf{R}L \to \mathsf{R}(L^{I})$ be an isomorphism. Consider $\tau\colon \Sigma L \to \mathsf{M}(\mathsf{R}L)$ and $\psi\colon \Sigma L^{I} \to \mathsf{M}(\mathsf{R}(L^{I}))$ to be the homeomorphism corresponding to L and L^{I} given in Remark 4.17 in [26]. It is clear that $\gamma\colon \mathsf{M}(\mathsf{R}L) \to \mathsf{M}(\mathsf{R}(L^{I}))$ with $\gamma(M_p) = M_{\gamma(p)}$ is one-one and onto. Hence $\psi^{-1}\gamma\tau\colon \Sigma L \to \Sigma L^{I}$ is a homeomorphism. $\square$

Chapter 6

Isocompactness in the category of locales

In this chapter we study isocompactness in pointfree topology. A space X is said tobe isocompact if every closed countably compact subset of X is compact. The class of isocompact spaces contains the class of compact spaces, the class of $\sigma-$compact spaces and the class of paracompact spaces, see [39]. The definition of isocompactness extends to locales easily by requiring that a locale to be isocompact if each of its countably compact closed sublocales is compact. Furthermore, this extension is conservative because closed sublocales are precisely its closed subspaces. That is to say, a space X is isocompact if and only if the locale $\mathbf{Lc}(X)$ of opens is isocompact.

6.1 Isocompact locales

Let $f : X \to Y$ be a continuous map of locales, and $f^* : \mathsf{O}Y \to \mathsf{O}X$ be the frame homomorphism defining it, and f_* for the right adjoint of this frame homomorphism. Further, for a frame L; we shall denote by c_{a_n}, the nucleus on L sending x to $a \vee x$.

Definition 6.1.1. A locale X is *countably compact* if every increasing countable cover of X has a finite subcover.

We omit the proof of the following lemma.

Lemma 6.1.1. *[23] A locale is countably compact if and only if every increasing countable cover contains the top element.*

The following definition extends easily from spaces.

Definition 6.1.2. A locale is *isocompact* if every countably compact closed sublocale is compact.

In [23], the authors observed that a closed sublocale of an isocompact locale is isocompact. To mimick their illustration, let X be a locale and take $a \in \mathsf{O}X$. Then the closed sublocales of the locale $\uparrow a$ are exactly the closed sublocales $\uparrow b$ of X for $b \geq a$.

Definition 6.1.3. A filter F in a locale X *clusters* if $\bigvee\{x^* \mid x \in F\} \neq 1$. F is *σ−fixed* if for any countable $S \subseteq F$, $\bigvee\{s^* \mid s \in S\} = 1$

In the next definition, X_b denotes the smallest dense sublocale of a locale X.

Definition 6.1.4. A locale X is said to be *almost realcompact* if any ideal in $\mathsf{O}(X_b)$ with $\bigvee_{\mathsf{O}X} I = 1$ has a countable subset S with $\bigvee_{\mathsf{O}X} S = 1$.

The proof of the following lemma can be found in [17], Proposition 3.4.

Lemma 6.1.2. *[23] A locale X is almost realcompact if and only if ever σ−fixed ultrafilter in $\mathsf{O}X$ clusters.*

Recall in Corollary 4.2.1 that a regular locale is compact if and only if every ultrafilter in it clusters. Now, the following lemma is culled out in [23], it is taken together with its proof.

Lemma 6.1.3. *[23] A regular almost realcompact, countably compact locale is compact.*

Proof. Let X be such a locale and suppose, by contradiction, that X is not compact. Then $\mathcal{O}X$ has an ultrafilter F which does not cluster. By the characterization of almost realcompact locales cited above, F is not $\sigma - fixed$, so there is a sequence $(x_n) \in F$ suchthat $\bigvee x_n^* = 1$. Thus $\{x_n^* \mid n \in \mathrm{N}\}$ is a countable cover of X. Since X is countably compact, there are finitely many indicies $n_1, n_2, \ldots, n_k$ such that $x_{n_1}^* \vee x_{n_2}^* \vee \ldots \vee x_{n_k}^* = 1$. Since F is an ultrafilter, it is prime, and so $x_{n_i}^* \in F$ for some i, which is a contradiction since F is a proper filter. Therefore X is compact. $\square$

Corollary 6.1.1. *[23] Every regular almost realcompact locale is isocompact.*

Proof. Let $\uparrow a$ be a countably compact closed sublocale of such a locale. By [17], $\uparrow a$ is almost realcompact, and hence, being regular, the lemma above implies that it is compact. $\square$

Next, we recall that a subspace S of a topological space X is said to be extension-closed if every open cover of S extends to an open cover of X (see [29]). This notion was extended to sublocales and to arbitrary continuous maps of locales in [23] as in the following definition. We will also need the following definition:

Definition 6.1.5. A continuous map of locales $f : X \to Y$ is *extension-closed* if $f_* : \mathcal{O}Y \to \mathcal{O}X$ preserves covers.

If a sublocale inclusion $j : S \hookrightarrow X$ is extension-closed, we shall also say the sublocale is extension-closed.

Definition 6.1.6. A continuous map of locales $f : X \to Y$ is said to be *perfect* if $f_* : \mathcal{O}Y \to \mathcal{O}X$ preserves directed joins. A *proper map* is a continuous map of locales which is closed and perfect.

For the purpose of our study we will need the following definition which was introduced in [23].

Definition 6.1.7. A continuous map of locales $f\colon X \to Y$ is *nearly perfect* if $f_*\colon \mathcal{O}Y \to \mathcal{O}X$ preserves directed covers.

Proposition 6.1.1. *[23] If $f: X \to Y$ is a nearly perfect localic surjection and X is isocompact, then Y is isocompact.*

Proof. Let $\uparrow a$ be a countably compact closed sublocale of Y. We show that the closed sublocale $\uparrow f^*(a)$ of X is countably compact. Let $\{b_n\}$ be an increasing countable cover of $\uparrow f^*(a)$. Then $f^*(a) \leq b_n$ for every n, so that $a \leq f_*(b_n)$ for every n. Since $\bigvee b_n = 1$ and the join is directed, we have $\bigvee f_*(b_n) = 1$, since f is nearly perfect, and so $\{f_*(b_n)\}$ is an increasing cover of the countably compact locale $\uparrow a$. Thus, $f_*(b_m) = 1$ for some index m, whence $b_m = 1$, implying that $\uparrow f^*(a)$ is countably compact. So $\uparrow f^*(a)$ is compact since X is isocompact. Now let C be a directed cover of $\uparrow a$. Then $\uparrow f^*[C]$ is a directed cover of $\uparrow f^*(a)$. By compactness of this locale, there is a $c \in C$ such that $f^*(c) = 1$. Since f is a localic surjection, $c = 1$, which implies that $\uparrow a$ is compact. Therefore Y is isocompact. ∎

If the codomain of the continuous map of locales is isocompact, we require a proper map to reflect the isocompactness of the domain

Proposition 6.1.2. *[23] If $f: X \to Y$ is a proper map of locales and Y is isocompact, then X is isocompact.*

Proof. Let $\uparrow b$ be a countably compact closed sublocale of X. We show that $\uparrow f_*(b)$ is countably compact. Let $\{a_n\}$ be an increasing cover of $\uparrow f_*(b)$. Then the set

$$\{b \vee f^*(a_n) \mid n = 1, 2, \ldots\}$$

is an increasing cover of $\uparrow b$. Since $\uparrow b$ is countably compact, there is $m \in \mathbb{N}$ such that $b \vee f^*(a_m) = 1$. Since f is a closed map, this implies that $f_*(b) \vee a_m = 1$, and hence $a_m = 1$ since $f_*(b) \leq a_m$. There $\uparrow f_*(b)$ is countably compact, and hence compact since Y is isocompact.

Now let C be a directed cover of $\uparrow b$, so that C is directed and $\bigvee C = 1$; whence $\bigvee\{f_*(c) \mid c \in C\} = 1$ because f_* preserves directed joins. Since $f_*(b) \leq f_*(c)$ for each $c \in C$, it follows that $f_*[C]$ is a directed cover of the compact locale $\uparrow f_*(b)$. Thus $f_*(c) = 1$ for some $c \in C$, which implies that

$1 = f^*f_*(c) \leq c$, whence it follows that $\uparrow b$ is compact. Therefore X is isocompact. $\square$

Bacon in [[4], Theorem 2.1] showed that if a space X is the union of a countable collection of closed isocompact subsets, then X is isocompact. This result has been extended to pointfree setting by Dube [23] as in the proposition below.

Proposition 6.1.3. *[23] A locale which is a join of countably many closed isocompact sublocales is isocompact.*

Proof. Let X be such a locale. The hypothesis, in frame terms, says there are countably many elements a_n in $\mathfrak{O}X$ such that $\uparrow a_n$ is isocompact for each n, and $\bigwedge_{n=1}^{\infty} c_{a_n} = id_{\mathfrak{O}X}$. Let $\uparrow b$ be a countably compact closed sublocale of X. Let C be a cover of $\uparrow b$. For each n, $\uparrow(a_n \vee b)$ is a closed sublocale of the countably compact locale $\uparrow b$, and is therefore countably compact. But now $\uparrow(a_n \vee b)$ is a countably compact closed sublocale of the isocompact locale $\uparrow a_n$, so it is compact. The set $\{a_n \vee b \vee c \mid c \in C\}$ is a cover of $\uparrow(a_n \vee b)$, so thereis a finite $C^{(n)} \subseteq C$ such that $(a_n \vee b) \vee \bigvee C^{(n)} = 1$. Now put $D = C^{(1)} \cup C^{(2)} \cup \ldots$, and observe that D is countable and $(a_n \vee b) \vee \bigvee D = 1$ for every n. Consequently, $\bigwedge_{n=1}^{\infty}(a_n \vee b \vee \bigvee D) = \bigwedge_{n=1}^{\infty}(a_n \vee (b \vee \bigvee D)) = 1$. But $\bigwedge_{n=1}^{\infty}(a_n \vee x) = x$ for each $x \in L$ since $\bigwedge_{n=1}^{\infty} c_{a_n} = id_L$, so $b \vee \bigvee D = 1$. Since $b \leq \bigvee D$ (Indeed, $b \leq d$ for each $d \in D$), it follows that $\bigvee D = 1$. Therefore D is a countable cover of the countably compact locale $\uparrow b$, so it has a finite subcover, which is a finite subcover extracted from C. Therefore $\uparrow b$ is compact, and hence X is isocompact. $\square$

The following corollary is apparent from the definition of an $F_\sigma-$sublocale and also extends [[4], Theorem 2.2] to pointfree setting.

Recall that an $F_\sigma-$sublocale of a locale is one which is expressible as a join of countably many closed sublocales.

Corollary 6.1.2. *[23] Every $F_\sigma-$sublocale of an isocompact locale is isocompact.*

Recall that a countable cover $\{a_n\}$ is shrinkable if there is a countable cover $\{b_n\}$ such that $b_n \prec a_n$ for each n. Recall also that a locale X is countably paracompact exactly if every countable cover $\{a_n\}$ is shrinkable. We have that dense extension-closed sublocales of isocompact countably paracompact locales are isocompact. This is seen from the proposition below culled from [23]. The proof is taken verbatim.

Proposition 6.1.4. *[23] A dense extension-closed sublocale of a countably paracompact isocompact locale is isocompact.*

Proof. Let $j : S \to X$ be a localic inclusion of a dense extension-closed sublocale of an isocompact locale. Let $\uparrow b$ be a countably compact closed sublocale of S. We show that $\uparrow j_*(b)$ is countably compact. Let $\{a_n\}$ be an increasing cover of $j_*(b)$. Since X is countably paracompact and $\{a_n\}$ is increasing cover of X, there is a cover $\{d_n\}$ of X such that $d_n \prec a_n$ for each n. The set $\{b \vee j^*(d_n) \mid n = 1, 2, \ldots\}$ is an increasing cover of $\uparrow b$, andso, by countable compactness of this locale, there is an index m such that $b \vee j^*(d_m) = 1$. Since j_* takes covers to covers, $j_*(b) \vee j_*j^*(d_n) = 1$, so that $j_*(b) \vee a_n = 1$ since j is dense and $d_m \prec a_m$. Thus, $a_m = 1$ because $j_*(b) \leq a_m$. Therefore $\uparrow j_*(b)$ is compact since X is isocompact. From this it is easy to deduce (using the fact that j_* takes covers to covers) that $\uparrow b$ is compact. ∎

6.2 Closure-Isocompactness

A variant of isocompactness called closure-isocompact was first considered in topological spaces by Sakai [36]. The concept of closure-isocompactness is stronger than the concept of isocompactness.

Definition 6.2.1. A locale is *closure-isocompact* (abbreviated *cl-isocompact*) if the closure of every countably compact complemented sublocale is compact. It is *fully cl-isocompact* if the closure of every countably compact sublocale is compact.

A locale X is said to be *perfectly normal* if it is normal and every element of $\mathfrak{O}X$ is a cozero element. Examples of perfectly normal locales include metrizable locales and Boolean locales.

Lemma 6.2.1. *[23] A locale which has a dense pseudocompact sublocale is pseudocompact.*

Proof. Let $j : S \to X$ be a dense pseudocompact sublocale of X. Let $\{a_n\}$ be a sequence in $\mathfrak{O}X$ with $a_1 \prec a_2, \ldots$ and $\bigvee a_n = 1$. Then $j^*(a_n) \prec j^*(a_{n+1})$ in $\mathfrak{O}S$ and $\bigvee j^*(a_n) = 1$. Then pseudocompactness of S yields an index n with $j^*(a_n) = 1$. By density of j,

$$1_{\mathfrak{O}X} = j_*j^*(a_n) \leq a_{n+1}. \text{ Therefore } X \text{ is pseudocompact.}$$

$\square$

For the next lemma, we recall from [23] that a sublocale S of a locale X is $\sigma-$compact if there are countably many compact sublocales K_n of X such that $S = \bigvee_n K_n$.

Lemma 6.2.2. *[23] A completely regular locale which is pseudocompact and σ-compact is compact.*

Proof. Let X be such a locale, and let $\{a_n\}$ be a sequence in $\mathfrak{O}X$ such that $\uparrow a_n$ is compact for each n and $\bigwedge_{n=1}^{\infty} \mathfrak{c}_{a_n} = id_{\mathfrak{O}X}$. Let C be a cover of X by cozero elements. By compactness of the frame $\uparrow a_n$, for each n there is a finite $B^{(n)} \subseteq C$ such that $a_n \vee \bigvee B^{(n)} = 1$. Put $B = \bigcup_{n=1}^{\infty} B^{(n)}$ and $b = \bigvee B$; then $a_n \vee b = 1$ for each n, and hence $\bigwedge_{n=1}^{\infty}(a_n \vee b) = 1$. Thus, $(\bigwedge_{n=1}^{\infty} \mathfrak{c}_{a_n})(b) = 1$, which implies that $\bigvee B = 1$. Since $B \subseteq Coz(\mathfrak{O}X)$ and is countable, the pseudocompactness of X implies that $\bigvee D = 1$, for some finite $D \subseteq B$. Thus, C has a finite subcover for X. By complete regularity, it follows that X is compact. $\square$

If $j : L \to L$ is a nucleus on a frame L and $a \in L$, then the join $j \vee c_a$ is the composite $j \circ c_a$ [15]. We will need this result in the proof of the proposition below.

Proposition 6.2.1. *[23] A complemented F_σ–sublocale of a completely regular cl-isocompact locale is cl-isocompact.*

Proof. Let Y be a complemented F_σ-sublocale of a cl-isocompact completely regular locale X. Write $Y = \bigvee_{n=1}^{\infty} K_n$, where K_n are the closed sublocales of X. Let Z be the complemented countably compact sublocale of Y. Then Z is a complemented sublocale of X, and hence $\overline{Z}$ is compact since X is cl-isocompact. We claim that each $Z \wedge K_n$ is countably compact. Fix any n, and say $\mathfrak{O}Z = Fix(z)$ and $\mathfrak{O}(K_n) = Fix(c_a)$, for some nucleus z on $\mathfrak{O}X$ and some $a \in \mathfrak{O}X$. Let $\{a_n\}$ be an increasing cover of $Fix(z \vee c_a)$. Then $(z(a \vee a_n))$ is an increasing cover of the countably compact frame $Fix(z)$, and hence thereis an index m such that $(z(a \vee a_m)) = 1$. But $(z(a \vee a_m)) = (z \vee c_a)(a_m) = a_m$; thus$Z \wedge K_n$ is countably compact. Since $Z \wedge K_n$ is a complemented sublocale of X and X is cl-isocompact, $\overline{Z \wedge K_n}$ is compact for each n. Because Z is complemented in $Sub(X)$, and $Z \leq \bigvee_n K_n$, we have

$$Z = Z \wedge \bigvee_n K_n = \bigvee_n (Z \wedge K_n).$$

Since, for each n, $Z \wedge K_n \leq \overline{Z \wedge K_n} \leq \overline{Z}$, the foregoing equality implies

$$Z \leq \bigvee_{n=1}^{\infty} \overline{Z \wedge K_n} \leq \overline{Z},$$

a consequence of which is that Z is dense in the sublocale $\bigvee_{n=1}^{\infty} \overline{Z \wedge K_n}$. Since Z is countably compact, and hence pseudocompact, it follows from Lemma 6.2.1 that $\bigvee_{n=1}^{\infty} \overline{Z \wedge K_n}$ is pseudocompact. But It is σ-compact; so, by Lemma 6.2.2, it is commpact, and hence, by [[32], Proposition III 1.2] it is a closed sublocale of X.

Now taking closure across the inequality $Z \leq \bigvee_{n=1}^{\infty} \overline{Z \wedge K_n} \leq \overline{Z}$ shows that $\overline{Z} = \bigvee_{n=1}^{\infty} \overline{Z \wedge K_n}$, whence $\overline{Z}$ is compact. Since each K_n is closed, we have $\overline{Z \wedge K_n} \leq \overline{K_n} = K_n$, and consequently $\overline{Z} = \bigvee_{n=1}^{\infty} \overline{Z \wedge K_n} \leq Y$, implying $\overline{Z}_Y = \overline{Z} \wedge Y = \overline{Z}$, hence $\overline{Z}_Y$ is compact. $\square$

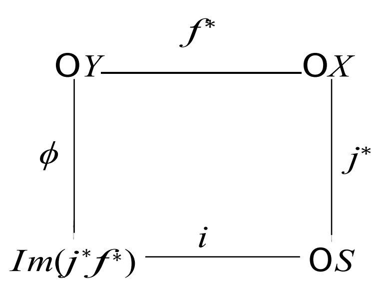
We show below that fully cl-isocompact locales are reflected by nearly perfect continuous maps. The proof is also taken verbatim.

Proposition 6.2.2. *[23] Let $f\colon X \to Y$ be a nearly perfect continuous map.*

1. *If Y is fully cl-isocompact, then so is X.*

2. *If Y is cl-isocompact, and f is a sublocale inclusion of a complemented sublocale, then X is cl-isocompact.*

Proof. (1): Let $j\colon S \to X$ be a sublocale inclusion with S countably compact. We then have the following commutative square

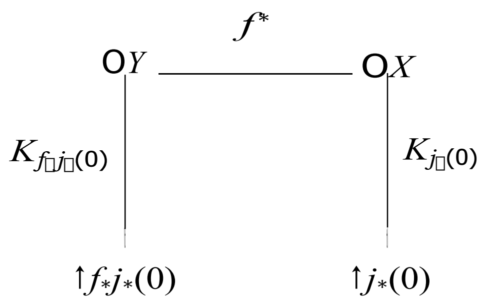


in **Frm**, where j is the inclusion map and ϕ maps as j^*f^*. We must show that $\uparrow j_*(0)$ is compact. Observe that $\phi_*(b) = f_*j_*(b)$ for every $b \in Im(j^*f^*)$. Since $Im(j^*f^*)$ is a subframe of $\mathsf{O}S$, it is isomorphic to the frame of opens of a countably compact sublocale of T. Therefore $\uparrow\phi_*(0)$ is compact because Y is fully cl-isocompact. Let D be a directed cover of $\uparrow\phi_*(0)$. Since f is nearly perfect, the set $\{f_*(d) \mid d \in D\}$ is a cover of Y, and hence the set $\{f_*(d) \vee j_*f_*(0) \mid d \in D\}$ is a directed cover of $\uparrow f_*j_*(0)$.

Therefore, by compactness of this frame, there is a $d \in D$ such that $f_*(d) \vee j_*f_*(0) = 1_{\mathcal{O}Y}$. Thus,

$1_{\mathcal{O}X} = f^*(f_*(d) \vee j_*f_*(0)) \leq d \vee j_*(0) = d$, Which shows that $\uparrow j_*(0)$ is compact.

(2): Let $j : S \to X$ be a sublocale inclusion with S complemented and countably compact. Using the diagram



as visual aid, we can complete the lines of the foregoing one, taking into account that S is complemented in $Sub(X)$.

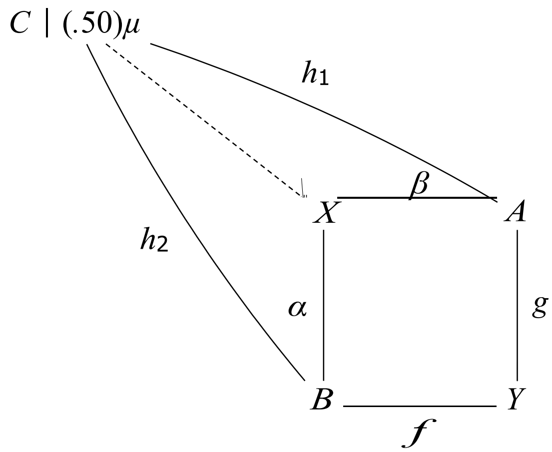
Corollary 6.2.1. *[23]A closed sublocale of a cl-isocompact locale is fully cl-isocompact.*

Next we recall the definition of a pullback in a category.

Definition 6.2.2. The *pullback* of a pair of morphisms (f, g) with $B \xrightarrow{f} Y \xleftarrow{g} A$ in a category C is a pair of morphisms (α, β) with $B \xleftarrow{\alpha} X \xrightarrow{\beta} A$ such that:

1. $g \circ \beta = f \circ \alpha$;

2. For any pair (h_1, h_2) of morphisms with $B \xleftarrow{h_2} C \xrightarrow{h_1} A$ and $g \circ h_1 = f \circ h_2$, there exists a unique morphism $\mu : C \to X$ such that $h_1 = \beta \circ \mu$ and $h_2 = \alpha \circ \mu$, as in the commutative diagram

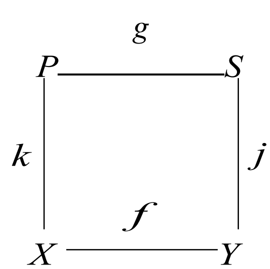


Proposition 6.2.3. *[23] Let $f: X \to Y$ be a proper surjection in* **Loc**.

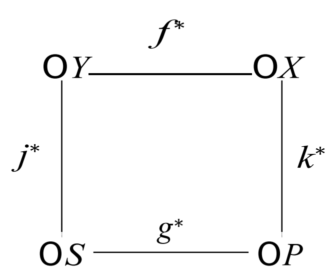
1. *If X is cl-isocompact, then Y is cl-isocompact.*

2. *If X is fully cl-isocompact, then X is fully cl-isocompact.*

Proof. 1. Let $j: S \to Y$ be a countably compact complemented sublocale of Y. Consider the squares



and



in **Loc** and **Frm**, respectively, where the one on the top depicts the pullback of f along j, and the on the bottom is its image under the functor O : **Loc** → **Frm**. By [[36], Proposition 4.2], g is a proper map, and, in fact, a locale surjection. Let (a_n) be an increasing cover of P. Then $\{g_*(a_n) \mid n \in \mathrm{N}\}$ is an increasing cover of S, and hence $g_*(a_n) = 1$, for some index n. Thus, $a_n = 1$, which implies that P is countably compact. But P is a sublocale of X (the "inverse image" of S under f), and, in fact, a complemented sublocale since inverse images of complemented sublocales are complemented. So, $\uparrow k_*(0)$ is compact because X is *cl*-isocompact. Let D be a directed cover of $\uparrow j_*(0)$. Then $\{f^*(d) \mid d \in D\}$ is a cover of X, and hence the set $\{k_*(0) \vee f^*(d) \mid d \in \mathrm{D}\}$ is a directed cover of the compact frame $\uparrow k_*(0)$, so that $k_*(0) \vee f^*(d) = 1$, for some $d \in D$. Since f is closed (as it is proper), we have $f_*k_*(0) \vee d = 1$. Now, $f \circ k = j \circ g$ implies that $f_*k_*(0) = j_*g_*(0) = j_*(0)$, the latter in viewof the fact that $g_*(0) = 0$, since g is a locale surjection. We deduce from $f_*k_*(0) \vee d = 1$ that $d = 1_X$ since $j_*(0) \leq d$. Thus, $\uparrow j_*(0)$ is compact, and hence Y is cl-isocompact. □

Chapter 7

Perfect compactifications of frames

In this chapter we study perfect compactifications of frames. Perfect compactifications of topological spaces were introduced by Sklyarenko [38] as a compactification Y of a space X having the property that $F_{r_Y} O\ U = Cl_Y F_{r_X} U$ for every open subset U of X. The set $O\ U = Y \setminus Cl_Y (X \setminus U)$ is the largest open subset of Y whose intersection with X gives the set U and F_r is the frontier (or boundary) operator. Perfect compactifications of frameswere introduced by Baboolal [2] in 2010.

7.1 Perfect compactifications

Baboolal introduced the perfect compactification of frames and rim-compact frames. The author defined Freudenthal compactifications of rim-compact frames. The Freudenthal compactification and Stone-Čech compactifications are examples of perfect compactifications. In this section, we study perfect compactifications in the context of pointfree topology.

Definition 7.1.1. Let $h : M \to L$ be a compactification of L, $r : L \to M$ be its right adjoint. Then (M, h) is said to be *perfect* with respect to an element $u \in L$ if $r(u \vee u^*) = r(u) \vee r(u^*)$. The compactification is said to be a *perfect compactification* of L if it is perfect with respect to every element of L.

The pointfree Stone-Čech compactification was introduced by Banaschewski and Mulvey in [14]. In [2], the author erroneously indicated that the proof of the following Theorem "follows from the corollary to Lemma 5 in [3]". There is no Lemma 5 in [3] and even Lemma 1.5 in [3] has no corollary. We have included the proof for this Theorem in our discussion.

Theorem 7.1.1. *[2] The Stone-Čech compactification of a completely regular frame is perfect.*

Proof. Let L be a completely regular frame and $h : M \to L$ be its Stone-Čech compactification with right adjoint $r : L \to M$. Take any $u \in L$, then $u \vee u^*$ is dense in L and since h is onto, we have $I, J \in M$ such that $h(I) = u$ and $h(J) = u^*$. Now

$$h(I \wedge J) = h(I) \wedge h(J) = u \wedge u^* = 0$$

and by denseness of h, $I \wedge J = 0$ in M. Now, $I \prec r(u)$ and $J \prec r(u^*)$ so that

$$\begin{aligned} r(u \vee u^*) &= \bigvee \{k \in M \mid h(k) \leq u \vee u^*\} \\ &\leq \bigvee (\{k \in M \mid h(k) \leq u\} \vee \{k \in M \mid h(k) \leq u^*\}) \\ &= \bigvee \{k \in M \mid h(k) \leq u\} \vee \bigvee \{k \in M \mid h(k) \leq u^*\} \\ &= r(u) \vee r(u^*). \end{aligned}$$

On the other hand, $u \vee u^* \geq u$ and $u \vee u^* \geq u^*$ so

$$r(u \vee u^*) \geq r(u) \text{ and } r(u \vee u^*) \geq r(u^*).$$

Therefore $r(u \vee u^*) \geq r(u) \vee r(u^*)$ and hence $r(u \vee u^*) = r(u) \vee r(u^*)$. Thus the Stone-Čech compactification is perfect. $\square$

The following lemma is due to Banaschewski (see [13]) and we take the proof verbatim.

Lemma 7.1.1. *[2] Let $h\colon M \to L$ be dense onto, with $r\colon L \to M$ its right adjoint. Then*

1. *$r(a^*) = r(a)^*$ for all $a \in L$,*
2. *$h(x^*) = h(x)^*$ for all $x \in M$.*

Proof. 1. $h\big(r(a) \wedge r(a^*)\big) = h\big(r(a)\big) \wedge h\big(r(a^*)\big) = a \wedge a^* = 0$ implies $r(a) \wedge r(a^*) = 0$ since h is dense, and hence $r(a^*) \leq r(a)^*$. Furthermore, $r(a) \wedge r(a)^* = 0$ imlpies $a \wedge h\big(r(a)^*\big) = 0$ which implies that $h\big(r(a)^*\big) \leq a^*$ and hence $r(a)^* \leq r(a^*)$.

2. $0 = x \wedge x^*$ implies $0 = h(0) = h(x) \wedge (x^*)$ which implies $h(x^*) \leq h(x)^*$. Furthermore, $r\big(h(x)^*\big) = r\big(h(x)\big)^*$ since $x \leq r\big(h(x)\big)$. Thus $hr\big(h(x)^*\big) \leq h(x^*)$, that is $h(x)^* \leq h(x^*)$.

$\square$

We shall say that in a frame L, the pair (u, v) *disconnects* w in L if $w = u \vee v$, $u \wedge v = 0$ and $u \neq 0$, $v \neq 0$. We then have the following theorem:

Theorem 7.1.2. *[2] The following are equivalent for a compactification $h\colon M \to L$ of L, r being the right adjoint of h.*

1. *$h\colon M \to L$ is a perfect compactification.*
2. *if a pair (u, v) disconnects w in L, then the pair $\big(r(u), r(v)\big)$ disconnects $r(w)$ in M.*
3. *r preserves disjoint binary joins.*

Proof. (1) $\Rightarrow$ (3): Take any $u, v \in L, u \wedge v = 0$. We shall show the non-trivial inequality $r(u \vee v) \leq r(u) \vee r(v)$. Now $u \wedge v = 0$ implies $v \leq u^*$ which implies $u \vee v \leq u \vee u^*$ and thus implying $r(u \vee v) \leq r(u \vee u^*) = r(u) \vee r(u^*)$. Similarly $r(u \vee v) \leq r(v) \vee r(v^*)$. Hence

$$
\begin{aligned}
r(u \vee v) &\leq \big(r(u) \vee r(u^*)\big) \wedge \big(r(v) \vee r(v^*)\big) \\
&= \big[\big(r(u) \vee r(u^*)\big) \wedge r(v)\big] \vee \big[\big(r(u) \vee r(u^*)\big) \wedge r(v^*)\big] \\
&= \big(r(u) \wedge r(v)\big) \vee \big(r(u^*) \wedge r(v)\big) \vee \big(r(u) \wedge r(v^*)\big) \vee \big(r(u^*) \wedge r(v^*)\big) \\
&= r(u \wedge v) \vee r(u^* \wedge v) \vee r(u \wedge v^*) \vee r(u^* \wedge v^*) \\
&= r(0) \vee r(v) \vee r(u) \vee r\big((u \vee v)^*\big) \\
&= 0 \vee r(v) \vee r(u) \vee \big(r(u \vee v)\big)^* \\
&= r(v) \vee r(u) \vee \big(r(u \vee v)\big)^*
\end{aligned}
$$

since h is dense and by virtue of Lemma 7.1.1 Thus $r(u \vee v) \leq r(u) \vee r(v)$ as required.

(3) $\Rightarrow$ (2): Suppose $w = u \vee v$, with $u \wedge v = 0, u \neq 0, v \neq 0$ in L. Then $r(w) = r(u) \vee r(v)$ with $r(u) \neq 0, r(v) \neq 0$ and $r(u) \wedge r(v) = r(u \wedge v) = r(0) = 0$. Thus $\big(r(u), r(u^*)\big)$ disconnects $r(w)$.

(2) $\Rightarrow$ (1): Take any $u \in L$. Let $w = u \vee u^*$. If either $u = 0$ or $u^* = 0$, then $r(u) = 0$ or $r(u^*) = 0$ by denseness of h, and the equality $r(u \vee u^*) = r(u) \vee r(u^*)$ must certainly hold. If $u \neq 0$ and $u^* \neq 0$, then (u, u^*) disconnects w, and thus the pair $\big(r(u), r(u^*)\big)$ disconnects $r(w)$. Hence $r(u \vee u^*) = r(u) \vee r(u^*)$. $\square$

Next we recall the concept of strong inclusion introduced by Banaschewski [6].

Definition 7.1.2. A *strong inclusion* on a frame L is a binary operation $\lhd$ on L such that:

1. if $x \leq a \lhd b \leq y$ then $x \lhd y$;

2. <J is a sublattice of $L \times L$;

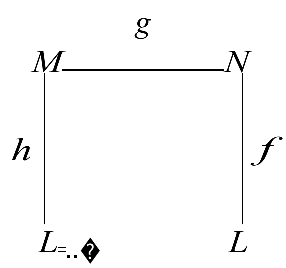
3. a <J $b \Longrightarrow a \prec b$;

4. a <J $b \Longrightarrow a$ <J c <J b for some $c \in L$;

5. a <J $b \Longrightarrow b^*$ <J a^*;

6. for each $a \in L$, $a = \bigvee\{x \in L \mid x$ <J $a\}$.

If a frame L is regular and continuous, then it has a smallest inclusion ◄ on L given by a ◄ b if and only if $a \prec b$ and either $\uparrow a^*$ or $\uparrow b$ is compact. Let $K(L)$ be the set of all compactifications of L, partially ordered by $(M, h) \leq (N, f)$ if and only if there exists a frame homomorphism $g : M \to N$ making the following diagram commute:



Also, let $S(L)$ be the set of all strong inclusions on L, partially ordered by set in- clusion. Banaschewski [6] shows that $K(L)$ is isomorphic to $S(L)$ by exhibiting maps $K(L) \to S(L)$ and $S(L) \to K(L)$ which are order preserving and inverses of each other. The map $K(L) \to S(L)$ is given as follows: For a compactification (M, h) of L, let $r : L \to M$ be the right adjoint of h. Then for any $x, y \in L$ define x <J y to mean that $r(x) \prec r(y)$. Then <J turns out to be a strong inclusion on L. For the map $S(L) \to K(L)$, let <J be any strong inclusion on L. Let γL be the set of all strongly regular ideals of L relative to <J, i.e. those ideals J of L for which $x \in J$ implies there exists $y \in J$ such that x <J y. Then $\bigvee : \gamma L \to L$ is dense, onto and γL is a regular subframe of $Idl(L)$, the

frame of ideals of L, so that $(\gamma L, \vee)$ is a compactification of L. This is the compactification associated with the given <J.

If (M, h) is a compactification of a frame L, it is of interest then to know what additional properties the associated strong inclusion must satisfy if (M, I) is a perfect compactification. This is given in the next

Proposition 7.1.1. *[2] Let* $h\colon M \to L$ *be a compactification of* L*, and* <J *the associated strong inclusion. If* (M, h) *is a perfect compactification, then* <J *satisfies*

for all $x, y \in L$, $x \leq y$, x <J $y \vee y^*$ *implies* x <J y.

Proof. Suppose $x \leq y$, and x <J $y \vee y^*$ for some $x, y \in L$. Then $r(x) \prec r(y \vee y^*) = r(y) \vee r(y^*)$, since (M, h) is a perfect compactification. Let $t \in L$ such that $r(x) \wedge t = 0$, and $t \vee r(y) \vee r(y^*) = 1$. Then

$$\begin{aligned} r(x) \wedge (t \vee r(y^*)) &= (r(x) \wedge t) \vee (r(x) \wedge r(y^*)) \\ &= 0 \vee r(x) \wedge r(y^*) \\ &= r(x) \wedge r(y^*) \\ &\leq r(x) \wedge r(x^*) \\ &= r(x \wedge x^*) \\ &= r(0) \\ &= 0, \text{ by the denseness of } h. \end{aligned}$$

Thus, $r(x) \prec r(y)$, with the separating element $t \vee r(y^*)$. Hence x <J y. $\square$

Proposition 7.1.2. *[2] Let* <J *be a strong inclusion on* L*, and* $(\gamma L, \vee)$ *the compactification associated with* <J*. If* <J *satisfies*

$x \leq y$, x <J $y \vee y^*$ *implies* x <J y *for all* $x, y \in L$

then $(\gamma L, \vee)$ *is a perfect compactification of* L.

Proof. We recall first from Banaschewski [6] that the right adjoint $k: L \to \gamma L$ of $\vee: \gamma L \to L$ is given by $k(a) = \{x \in L \mid x <J a\}$. We have to show that $k(a \vee a^*) = k(a) \vee k(a^*)$ forany $a \in L$.

Suppose that $x \in k(a \vee a^*)$, then $x <J a \vee a^*$. Further, $x = (x \wedge a) \vee (x \wedge a^*)$. Now $x \wedge a \leq a, x \wedge a \leq x <J a \vee a^*$ implies $x \wedge a <J a \vee a^*$, which by virtue of the condition satisfied by $<J$ implies $x \wedge a <J a$. Furthermore $x <J a \vee a^*$ implies $x <J a^* \vee a^{**}$, since $a \leq a^{**}$. Hence $x \wedge a^* <J a^* \vee a^{**}$. Since $x \wedge a^* \leq a^*$, by the condition satisfied by $<J$ again, we have$x \wedge a^* <J a^*$. Thus $x \in k(a) \vee k(a^*)$. The reverse inclusion being clear, this proves that$k(a \vee a^*) = k(a) \vee k(a^*)$.

In view of the isomorphism between $K(L)$ and $S(L)$ mentioned above, the two propositions above imply the following:

Proposition 7.1.3. *[2] A compactification (M, h) of a frame L is perfect if and only if its associated strong inclusion $<J$ satisfies*

$$x \leq y, x <J y \vee y^* \text{ implies } x <J y \text{ for all } x, y \in L.$$

Remark 7.1.1. [2] The above proof shows in effect that (M, h) is perfect with respect to $y \in L$ if and only if whenever $x \leq y, x <J y \vee y^*$ then $x <J y$.

Given an arbitrary compactification (M, h) of L, we do not in general expect its right adjoint r to preserve disjoint binary joins. However, if elements $u, v \in L$ are not just disjoint but such that $u <J v^*$ then $r(u \vee u) = r(u) \vee r(v)$ always holds as we show below. Of course we also end this section with the following proposition.

Proposition 7.1.4. *[2] Let $h : M \to L$ be a compactification of L, $<J$ the induced strong inclusion. If $u, v \in L$ and $u <J v^*$ then $r(u \vee v) = r(u) \vee r(v)$.*

Proof. Since $u \prec\!\!\prec v^*$, we have $r(u) \prec r(v^*)$ and hence $r(u)^* \vee r(v)^* = 1$. Thus $r(u^*) \vee r(v^*) = 1$, since $r(u)^* = r(u^*)$ by Lemma 7.1.1. Hence

$$\begin{aligned} r(u \vee v) &= r(u \vee v) \wedge \big(r(u^*) \vee r(v^*)\big) \\ &= \big(r(u \vee v) \wedge r(u^*)\big) \vee \big(r(u \vee v) \wedge r(v^*)\big) \\ &= r\big((u \vee v) \wedge u^*\big) \vee r\big((u \vee v) \wedge v^*\big) \\ &= r(v \wedge u^*) \vee r(u \wedge v^*) \\ &\leq r(u) \vee r(v) \end{aligned}$$

proving the non-trivial inequality. ∎

7.2 Rim-compact frames and Freudenthal compactification

Rim-compact spaces (also called peripherally (bi) compact spaces) are Hausdorff topological spaces having a basis for the topology consisting of open sets with compact frontiers (see [38]). Let X be a topological space. For $U \in \mathfrak{O}X$, $Fr_X U = Cl_X U \setminus U$. Now for any $V \in \mathfrak{O}X$, $\uparrow V \cong \mathfrak{O}(X \setminus V)$ as frames. Thus we have that $\uparrow U \cup U^*$ is compact if and only if $\mathfrak{O}\big(X \setminus (U \cup U^*)\big)$ is compact if and only if $\mathfrak{O}\big((X \setminus U) \cap (X \setminus U^*)\big)$ is compact if and only if $\mathfrak{O}\big((X \setminus U) \cap Cl_X U\big)$ is compact if and only if $\mathfrak{O}(Fr_X U)$ is compact if and only if $Fr_X U$ is compact. Therefore X is *rim-compact* as a topological space if and only if $\mathfrak{O}(X)$ is *rim-compact* as a frame.

Definition 7.2.1. A regular frame L is called *rim-compact* if for each $a \in L$, $a = \bigvee\{u \in L \mid \uparrow(u \vee u^*) \text{ is compact}\}$.

In [2] the author remarked that a topological space X is rim-compact if and only if the lattice $\mathsf{O}X$ of its open subsets is rim-compact.

The functors Σ and O induce a dual equivalence between the category of spatial frames and the category of sober topological spaces. Since every rim-compact space is sober and the lattice $\mathsf{O}X$ is rim-compact for each space X, it follows that the functor O embeds the category of rim-compact spaces into the category of rim-compact frames. Therefore the category of rim-compact frames is a wider class than the category of rim-compact spaces.

Definition 7.2.2. Let L be a rim-compact frame. A π-*compact* basis B for L is a basis B for L such that:

1. $a \in B$ implies $\uparrow(a \vee a^*)$ is compact;
2. $a \in B$ implies $a^* \in B$;
3. $a, b \in B$ implies $a \wedge b, a \vee b \in B$.

Example 7.2.1. [2] Let L be a rim-compact frame. Observe that L always has at least one $\pi-$compact basis: Indeed, let B be the basis for L consisting of all elements b such that $\uparrow(b \vee b^*)$ is compact. We have to show (2) and (3) in the above definition. Let $a \in B$. Since $a \vee a^* \leq a^* \vee a^{**}$, we have $\uparrow(a^* \vee a^{**})$ is compact since $\uparrow(a \vee a^*)$ is compact andthis proves (2). For (3) let $a, b \in B$, then we have to show that $\uparrow\big((a \wedge b) \vee (a \wedge b)^*\big)$ and $\uparrow\big((a \vee b) \vee (a \vee b)^*\big)$ are compact. By Lemma 2.1.1 $\uparrow a$ and $\uparrow b$ are compact if and only if $\uparrow(a \wedge b)$ is compact. Now, $(a \wedge b) \vee (a \wedge b)^* = \big(a \vee (a \wedge b)^*\big) \wedge \big(b \vee (a \wedge b)^*\big) \geq (a \vee a^*) \wedge (b \vee b^*)$.Hence $\uparrow\big((a \wedge b) \vee (a \wedge b)^*\big)$ is compact since $\uparrow\big((a \vee a^*) \wedge (b \vee b^*)\big)$ is compact by the above note. Also

$$
\begin{aligned}
(a \vee b) \vee (a \vee b)^* &= (a \vee b) \vee (a^* \wedge b^*) \\
&= (a \vee b \vee a^*) \wedge (a \vee b \vee b^*) \\
&\geq (a \vee a^*) \wedge (b \vee b^*)
\end{aligned}
$$

and so $\uparrow\big((a \vee b) \vee (a \vee b)^*\big)$ is also compact.

Lemma 7.2.1. *[2] Let L be a rim-compact frame and B be a π-compact basis for L. If $w \in L$ and $u \in B$ with $w \vee u = 1$, then there exists $v \in B$ such that $v \prec u$ and $w \vee v = 1$.*

Proof. Using regularity and the fact that B is a basis for L, we have $w = \bigvee\{x \in L \mid x \prec w, w \in B\}$. Then $u \vee \bigvee\{x \in L \mid x \prec w, x \in B\} = 1$ and hence $u \vee u^* \vee \bigvee\{x \in L \mid x \prec w, x \in B\} = 1$. Since $\uparrow(u \vee u^*)$ is compact, we can find $x_i \in B, x_i \prec w$ for $i = 1, 2, \ldots, n$ such that $u \vee u^* \vee \bigvee_{i=1}^{n} x_i = 1$. Put $x = \bigvee_{i=1}^{n} x_i$, we have $x \in B, x \prec w$ and $u \vee u^* \vee x = 1$. Let $v = u \wedge x^*$. Then $v \in B$, and furthermore

$$\begin{aligned} w \vee v &= w \vee (u \wedge x^*) \\ &= (w \vee u) \wedge (w \vee x^*) \\ &= 1 \wedge 1 \\ &= 1. \end{aligned}$$

Also, $v \prec u$:

$$\begin{aligned} v \wedge (u^* \vee x) &= (v \wedge u^*) \vee (v \wedge x) \\ &= (u \wedge x^* \wedge u^*) \vee (v \wedge x) \\ &= 0 \text{ and } u \vee (u^* \vee x) = 1. \end{aligned}$$

$\square$

Proposition 7.2.1. *[2] Let B be a π-compact basis for a rim-compact frame L. Define $\lhd$ on L by : $a \lhd b \Leftrightarrow$ there exists $u \in B$ such that $a \prec u \prec b$. Then $\lhd$ is a strong inclusion on L.*

Proof. 1. $x \leq a \lhd b \leq y \Rightarrow x \lhd y$: Find $u \in B$ such that $a \prec u \prec b$. Then, of course, $x \prec u \prec y$ and so $x \lhd y$.

2. $\lhd$ is a sublattice of $L \times L$: Condition 3 together with 2 of the definition of $\pi-$compact basis gives us $0, 1 \in B$, and then of course $0 \lhd 0$, $1 \lhd 1$. Furthermore, the implications $x \lhd a, b$ implies $x \lhd a \wedge b$, and $x, y \lhd a$ implies $x \vee y \lhd a$ follow from the properties of rather below relation $\prec$ and the fact that B is closed under finite meet and finite joins.

3. $x \lhd a$ implies $x \prec a$ trivially.

4. Now suppose $x \lhd y$. Then there exists $u \in B$ such that $x \prec u \prec y$. Now $x^* \vee u = 1$, and so by Lemma 7.2.1, there exists $v \in B, v \prec u$ such that $x^* \vee v = 1$. Hence $x \prec v \prec u \prec y$. Similarly we can get $w \in B$ such that $x \prec v \prec w \prec u \prec y$. Thus $x \lhd w \lhd y$.

5. Also, $x \lhd a$ implies $a^* \lhd x^*$ follows from the properties of $\prec$ and the fact that B is closed under pseudocomplementation.

6. Now for any $a \in L, a = \bigvee\{x \in L \mid x \prec a, x \in B\}$. For $x \in B$ and $x \prec a$ we have $x^* \in B$ and $a \vee x^* = 1$. By Lemma 7.2.1, there exists $v \in B$, $v \prec x^*$ and $a \vee v = 1$. Hence $x \prec v^* \prec a$ with $v^* \in B$. Thus $x \lhd a$ and $a = \bigvee\{x \in L \mid x \lhd a\}$.

$\square$

Let L be any rim-compact frame, and let B be any π-compact basis for L. Let $\gamma_B L$ denote the compactification of L associated with the strong inclusion $\lhd_B$ given as in Proposition 7.2.1, that is, $a \lhd_B b \Leftrightarrow$ there exists $u \in B$ such that $a \prec u \prec b$. We then have the following.

Proposition 7.2.2. *[2] Let $\gamma_B L$ be the compactification associated with the $\pi-$compact basis B of a rim-compact frame L, and let (M, h) be any compactification of L such that $(\gamma_B L, \vee) \leq (M, h)$. Then (M, h) is perfect with respect to every element of B.*

Proof. By remark 7.1.1 we have to show that for each $u \in B$, whenever $x \leq u$ and $x \lhd_J u \vee u^*$, then $x \lhd_J u$. Here $\lhd_J$ is induced by (M, h). For $x \leq u$, $x \lhd_J u \vee u^*$, we have $x \prec u \vee u^*$ and thus $x^* \vee u \vee u^* = 1$. Hence $x^* \vee u = 1$, since $x \leq u$. By Lemma 7.2.1 there exists $v \in B$, $v \prec u$ such that $x^* \vee v = 1$. Then $x \prec v \prec u$ with $v \in B$. Thus $x \lhd_{J_B} u$, and hence $x \lhd_J u$, since $(\gamma_B L, \vee) \leq (M, h)$. ∎

The author in [2] remarked that $(\gamma_B L, \vee)$ is a compactification of a rim-compact frame L which is perfect with respect to every element of B. But it need not be perfect with respect to every element of L, and consequently need not be a perfect compactification. Call a compactification (M, h) of a rim-compact frame L a *π-compactification* of L if there exists a $\pi-$compact basis B of L such that $(M, h) \cong (\gamma_B L, \vee)$. We show in the next proposition that such compactification of L possesses a base intimately connected with the given $\pi-$compact base for L.

Proposition 7.2.3. *[2] Let $(\gamma_B L, \vee)$ be the $\pi-$compactification of the rim-compact frame with $\pi-$compact basis B. Let $k\colon L \to \gamma_B L$ be the right adjoint of $\vee\colon \gamma_B L \to L$, that is, $k(a) = \{x \in L \mid x \lhd_{J_B} a\}$. Then $k(B) = \{k(u) \mid u \in B\}$ is a basis for $\gamma_B L$.*

Proof. For any $J \in \gamma_B L$, $J = \bigcup \{k(a) \mid a \in J\}$. Since B is a basis for L, we have for each $a \in J$, $a = \bigvee \{u \in B \mid u \leq a\}$. We shall show that $k(a) = \bigvee \{k(u) \mid u \in B, u \leq a\}$. Let $x \in k(a)$. Then $x \lhd_{J_B} a$ and thus, since $\lhd_J$ interpolates, there exists $c \in L$ such that $x \lhd_{J_B} c \lhd_{J_B} a$. Hence we can find $u, v \in B$ such that $x \prec u \prec c \prec v \prec a$. Thus $k(a) = \bigvee \{k(u) \mid u \in B, u \leq a\}$. ∎

Following the discussion before Proposition 7.2.3, it would be nice if there were a π-compact basis B for a rim-compact frame L for which $\gamma_B L$ is perfect with respect to every element of L and not just to those elements in B. This is indeed the case, as we show below, if we take B to consist of the totality of all elements u of L such that $\uparrow(u \vee u^*)$ is compact.

Denote this compactification with the above mentioned basis by $(\gamma L, \vee)$. We call this the *Freudenthal compactification* of the rim-compact frame L. Much interesting is that this compactification is perfect as shown below.

Proposition 7.2.4. *[2] The Freudenthal compactification γL is perfect.*

Proof. Let $u \in L$ be arbitrary, $x \leq u$, x <J $u \vee u^*$, where <J is the strong inclusion associated with B mentioned above. We must show x <J u. Now find v such that $\uparrow(u \vee v^*)$ is compact and $x \prec v \prec u \vee u^*$. Let $w = v \wedge u$. Then $x \prec w$. Indeed $x \leq u, x \prec u \vee u^*$ implies $x^* \vee u \vee u^* = 1$ and hence $x^* \vee u = 1$. Thus $x \prec u$, and since $x \prec v$ as well, we have $x \prec u \wedge v = w$. Furthermore $w \prec u$: Find t such that $v \wedge t = 0, t \vee u \vee u^* = 1$. Then $w \wedge (t \vee u^*) = (w \wedge t) \vee (w \wedge u^*) = (v \wedge u \wedge t) \vee (v \wedge u \wedge u^*) = 0$. Thus $w \prec u$, with a separating element $t \vee u^*$. We claim that $v \vee v^* \leq w \vee w^*$: Clearly $v^* \leq w^*$ since $w \leq v$. Hence $v^* \leq w \vee w^*$. Also, $v \prec u \vee u^*$ implies $v = (v \wedge u) \vee (v \wedge u^*) = w \vee (v \wedge u^*) \leq w \vee u^* \leq w \vee w^*$. Thus $v \vee v^* \leq w \vee w^*$ and hence $\uparrow(w \vee w^*)$ is compact. $\square$

7.3 The two point compactification

In this section we study a compactification for a class of regular continuous frames constructed by Baboolal [3]. This is the analog of the two point compactification for locally compact Hausdorff spaces.

Take L to be a regular continuous frame. Suppose L has elements u and v with the property that $u \wedge v = 0$, $\uparrow(u \vee v)$ is compact, but neither $\uparrow u$ nor $\uparrow v$ is compact. Put

$$N_1 = \{x \in L \mid \uparrow(x \vee u) \text{ is compact}\} \text{ and } N_2 = \{x \in L \mid \uparrow(x \vee v) \text{ is compact}\}.$$

Proposition 7.3.1. *[3] Let L be a regular continuous frame and $a \in L$. Then $a \ll 1$ if and only if $\uparrow a^*$ is compact.*

Proof. (⇒): Suppose $a \ll 1$ and $1 = \bigvee C$ for some $C \subset L$ with $a^* \leq b$ for each $b \in C$. Since the relation $\ll$ is interpolation, we can find $x \in L$ such that $a \ll x \ll 1$. But $x \ll 1$ implies there exists finite $K \subseteq C$ such that $x \leq \bigvee K$. Now since $a^* \vee x = 1$, we have $a^* \vee \bigvee K = 1$ and hence $\bigvee K = 1$. Thus $\uparrow a^*$ is compact.

(⇐): Suppose $\uparrow a^*$ is compact and let $\bigvee C = 1$ for some $C \subseteq L$. Then $\bigvee \{a^* \vee x \mid x \in C\} = 1$ and by compactness of $\uparrow a^*$, we have $1 = a^* \vee \bigvee K$ for some finite subset $K \subseteq C$. Thus $a \leq \bigvee K$, and hence $a \ll 1$. ∎

Recall that a filter F in a frame L is said to be regular if $x \in F$ implies that there exists $y \in F$ such that $x \prec y$. We show below that N_1 and N_2 are regular proper filters of L.

Lemma 7.3.1. *[3] N_1 and N_2 are regular proper filters of L.*

Proof. Note that $0 \not\in N_1, N_2$ since $\uparrow(0 \vee u) = \uparrow u$ and $\uparrow(0 \vee v) = \uparrow v$ are not compact. Let $x, y \in N_1$, therefore $\uparrow(x \vee u)$ and $\uparrow(y \vee u)$ are compact, and thus $\uparrow(x \vee u) \wedge \uparrow(y \vee u)$ is compact. But

$$\uparrow(x \vee u) \wedge \uparrow(y \vee u) = \uparrow\big((x \vee u) \wedge (y \vee u)\big) = \uparrow\big((x \wedge y) \vee u\big) \text{ is compact.}$$

Hence $x \wedge y \in N_1$. Now let $x \in N_1$ and $x \leq y$ for some $y \in L$. Then $x \vee u \leq y \vee u$ implies $\uparrow(y \vee u) \subseteq \uparrow(x \vee u)$ and by compactness of $\uparrow(x \vee u)$, we have $\uparrow(y \vee u)$ is compact and hence $y \in N_1$. Using the same argument for N_2 shows that N_1 and N_2 are proper filters of L. To show regularity, let $a \in N_1$. Now $1 = \bigvee\{x \in L \mid x \ll 1\}$, so $\uparrow(a \vee u)$ compact implies there exists $x \ll 1$ such that $a \vee u \vee x = 1$. But $x \ll 1$ if and only if $\uparrow x^*$ is compact and thus $x^* \in N_1$.

Now, $x \vee u \vee a = 1$ implies $(x \vee u)^{**} \vee a = 1$ which implies $(x \vee u)^* \prec a$ and hence implying $x^* \wedge u^* \prec a$. Since $u \wedge v = 0$ implies $v \leq u^*$, and since $v \in N_1$, we have $u^* \in N_1$. Thus $x^* \in N_1$, $u^* \in N_1$ and hence $x^* \wedge u^* \in N_1$. Thus N_1 and N_2 are regular. ∎

Lemma 7.3.2. *[3] For any $a \in L$, the up-set $\uparrow a$ is compact if and only if $a \in N_1 \cap N_2$.*

Proof. ($\Rightarrow$): Let $a \in L$ and suppose $\uparrow a$ is compact. Then $\uparrow(a \vee u)$ and $\uparrow(a \vee v)$ are compact and hence $a \in N_1 \cap N_2$.
($\Leftarrow$): If $a \in N_1 \cap N_2$, then $\uparrow(a \vee u)$ and $\uparrow(a \vee v)$ are compact. Hence

$$\uparrow(a \vee u) \wedge (a \vee v) = \uparrow\big((a \vee u) \wedge (a \vee v)\big) = \uparrow(a \vee u \wedge v) = \uparrow a \text{ is compact.}$$

$\square$

Now define $a <J\, b$ in L by : $a <J\, b \Leftrightarrow a \prec b$ and for each $i = 1, 2$ either $a^* \in N_i$ or $b \in N_i$.

Lemma 7.3.3. *[3] The relation $<J$ is a strong inclusion on L.*

Proof. 1. Assume $x \leq a <J\, b \leq y$. Now $a \prec b$, so clearly $x \prec y$. Take any N_i ($i = 1, 2$).If $a^* \in N_i$, then $x^* \in N_i$ since $a^* \leq x^*$ and N_i is a filter. If $b \in N_i$, then $b \leq y$ impliesy $y \in N_i$. Thus $x <J\, y$.

2. $0 <J\, 0$ since $0 \prec 0$ and $0^* = 1 \in N_i$ for each i. Also $1 <J\, 1$ since $1 \prec 1$ and $1 \in N_i$. Then suppose $x, y <J\, a$. Then $x \prec a$, $y \prec a$, so $x \vee y \prec a$. Fix i. If $a \in N_i$, then$x \vee y <J\, a$. If $x^* \in N_i, y^* \in N_i$, then $x^* \wedge y^* \in N_i$, that is, $(x \vee y)^* \in N_i$. Thus$x \vee y <J\, a$. Now suppose $x <J\, a$, $x <J\, b$. Then $x \prec a$, $x \prec b$, so $x \prec a \wedge b$. Fix i. If$a \wedge b \in N_i$, then $x <J\, a \wedge b$. If not, then either a lies outside N_i or b lies outside N_i. Thus $x^* \in N_i$ and hence $x <J\, a \wedge b$.

3. $x <J\, a$ implies $x \prec a$ follows from the definition.

4. Now suppose $x <J\, a$. Then $x \prec a$. If either $\uparrow x^*$ or $\uparrow a$ is compact, then $x \blacktriangleleft a$ in which case there exists $y \in L$ such that $x \blacktriangleleft y \blacktriangleleft a$. Since $\blacktriangleleft \subseteq <J$ this means $x <J\, y <J\, a$ so that interpolation holds. If $\uparrow x^*$ is not compact and $\uparrow a$ is not compact then both x^* and a lie outside $N_1 \cap N_2$ by the above lemma.

There are two cases here : (a) $x^* \in N_1$ and $a \in N_2$ and (b) $x^* \in N_2$ and $a \in N_1$. Symmetry considerations make it sufficient to consider just one of these cases, say $x^* \in N_1$ and $a \in N_2$. We seek y such that $x <_J y <_J a$. By the fact that $\uparrow(x^* \vee u)$ is compact, $x \prec a$ and $a = \bigvee\{z \in L \mid z \ll a\}$, we can find $z \ll a$ such that $x^* \vee u \vee z = 1$. Also from the fact that $\uparrow(a \vee v)$ is compact, $x \prec a$ and $x^* = \bigvee\{t \in L \mid t \ll x^*\}$, we can find $t \ll x^*$ such that $a \vee v \vee t = 1$. Now we have $x \prec u \vee z$ and $x \prec t^*$, and thus $x \prec (u \vee z) \wedge t^* = (u \wedge t^*) \wedge (z \wedge t^*)$. We also have $u \wedge t^* \prec a$, since the element $u \vee t$ is such that $a \vee v \vee t = 1$ and

$$(u \wedge t^*) \wedge (v \vee t) = (u \wedge t^* \wedge v) \vee (u \wedge t^* \wedge t) = 0.$$

Also $z \wedge t^* \leq z \prec a$. Hence $(u \wedge t^*) \vee (z \vee t^*) \prec a$. Thus

$$x \prec (u \wedge t^*) \vee (z \wedge t^*) = t^* \wedge (u \vee z) \prec a.$$

Put $y = t^* \wedge (u \vee z)$. We claim that $x <_J y <_J a$.

$x <_J y$: Obviously $x \prec y$, $x^* \in N_1$. We show that $y \in N_2$. Now $t \ll 1$ implies $\uparrow t^*$ is compact and hence $t^* \in N_1 \cap N_2$ by the above lemma. Also $u \vee z \in N_2$ since $u \in N_2$. Thus $t^* \wedge (u \vee z) \in N_2$, that is, $y \in N_2$. Hence $x <_J y$.

$y <_J a$: Obviously $y \prec a$ and $a \in N_2$. Now $z \ll a$ implies $z \ll 1$ and so $\uparrow z^*$ is compact. Hence $z^* \in N_1$. Also $v \in N_1$, $v \ll u^*$ implies $u^* \in N_1$. Furthermore, $z^* \wedge u^* \ll y^*$ since

$$z^* \wedge u^* \wedge y = z^* \wedge u^* \wedge (t^* \wedge (u \vee z)) = (z^* \wedge u^* \wedge t^* \wedge u) \vee (z^* \wedge u^* \wedge t^* \wedge z) = 0.$$

Hence $y^* \in N_1$ and thus $y <_J a$. Thus $<_J$ interpolates.

5. $x <_J a \Longrightarrow a^* <_J x^*$: $x <_J a$ implies $x \prec a$ and hence $a^* \prec x^*$. Again if either $\uparrow x^*$ or $\uparrow a$ is compact, then $a \blacktriangleleft a$ and hence $a^* \blacktriangleleft x^*$ from which $a^* <_J x^*$. As in 4 we need

only consider the case $x^* \in N_1$, $a \in N_2$. In this case $a \leq a^{**}$ implies $a^{**} \in N_2$. Since also $x^* \in N_1$ we have then that $a^* <_J x^*$ as required.

6. For each $a \in L$, $a = \bigvee \{x \in L \mid x \blacktriangleleft a\}$. But as remarked earlier, $x \blacktriangleleft a$ implies $x <_J a$. Hence $a = \bigvee \{x \in L \mid x <_J a\}$.

Thus $<_J$ is a strong inclusion on L. $\square$

Recall that a *congruence* on a frame L is an equivalence relation on L which is also a subframe of $L \times L$. The congruence lattice $\mathsf{C}L$ of L consists of all the congruences on L.

Definition 7.3.1. For any compactification $h : M \to L$, the *remainder* of L in the compactification is the quotient M/Θ where $\Theta = (ker\ \ h)^*$, the pseudocomplement of $ker\ \ h$ in the congruence lattice $\mathsf{C}L$ of M.

Let $\vee : \alpha L \to L$ be the least compactification corresponding to the least strong inclusion $\blacktriangleleft$ on L and let $k : L \to \alpha L$ be its right adjoint. Let $J = \bigvee \{k(x) \in \alpha L \mid x \ll 1\}$, so thatby Section 2 in [3], $\uparrow J$ would be the remainder of L in αL. Note further that $J < L$. We then have the following:

Lemma 7.3.4. *[3] If $w \in L$, then $\uparrow w$ is compact if and only if $k(w) \vee J = L$.*

Proof. ($\Rightarrow$): Assume $\uparrow w$ is compact. Now $w \vee \bigvee \{x \in L \mid x \ll 1\} = 1$. Thus $\bigvee \{w \vee x \mid x \ll 1\} = 1$, so by compactness of $\uparrow w$ we have $w \vee x = 1$ for some $x \ll 1$. Thus $w \vee x^{**} = 1$, whence $x^* \prec w$. Since $\uparrow w$ is compact, this means $x^* \blacktriangleleft w$ and hence $x^* \in k(w)$. Now $\bigvee J = 1$ implies $\bigvee \{x^* \vee y \mid y \in J\} = 1$, from which, since $\uparrow x^*$ is compact (as $x \ll 1$), we have $x^* \vee y = 1$ for some $y \in J$. Thus $k(w) \vee J = L$.

($\Leftarrow$): Assume $k(w) \vee J = L$. Then $1 = x \vee y$ for some $x \blacktriangleleft w$, $y \in J$. Now either $\uparrow x^*$ is compact or $\uparrow w$ is compact. If $\uparrow x^*$ is compact, then $x \ll 1$, so that $x \ll z \ll 1$ for some $z \in L$. Hence $x \blacktriangleleft z \ll 1$ so that $x \in k(z) \subseteq J$. Thus $1 = x \vee y \in J$ which would imply $J = L$, a contradiction. Thus $\uparrow w$ is compact. $\square$

As remarked in [3], the necessity of the above lemma was observed that $\uparrow w$ compact implies $k(w) \vee J = L$ is in fact true for any strong inclusion $\lhd_J$ on L, whether least or not, where as before k is the right adjoint of the join map and $J = \bigvee\{k(x) \in \alpha L \mid x \ll 1\}$.

Theorem 7.3.1. *[3] Let L be regular continuous. Suppose L has elements u and v with the property that $u \wedge v = 0$, $\uparrow(u \vee v)$ is compact, but neither $\uparrow u$ nor $\uparrow v$ is compact. Let $N_1 = \{x \in L \mid \uparrow(x \vee u)$ is compact$\}$ and $N_2 = \{x \in L \mid \uparrow(x \vee v)$ is compact$\}$. The compactification $\vee\colon \gamma L \to L$ arising from the strong inclusion $\lhd_J$ given by: $a \lhd_J b \Leftrightarrow a \prec b$ and for each $i = 1, 2$ either $a^* \in N_i$ or $b \in N_i$ is such that the remainder of L in it is disconnected.*

Proof. Let $J = \bigvee\{k(x) \in \alpha L \mid x \ll 1\}$ where $k : L \to \gamma L$ is the right adjoint of the join map. We show that the remainder $\uparrow J$ in γL is disconnected. We claim that $k(u \vee v) = k(u) \vee k(v)$: For this, obviously $k(u) \vee k(v) \subseteq k(u \vee v)$. For the reverse, take $s \in k(u \vee v)$.Then $s \lhd_J u \vee v$ and hence $s \prec u \vee v$. Since $u \wedge v = 0$ we have that $s \wedge u \prec u$ and $s \wedge v \prec v$. We have $(s \wedge u)^* \vee u = 1$, so $\uparrow(s \wedge u)^* \vee u = \uparrow 1$ is compact. Thus $(s \wedge u)^* \in N_1$. Also, since $\uparrow(u \wedge v)$ is compact, we have $u \in N_2$. Thus for each i we have either $(s \wedge u)^* \in N_i$ or $u \in N_i$, that is, $s \wedge u \lhd_J u$. Similarly $s \wedge v \lhd_J v$. Thus
$s = (s \wedge u) \vee (s \wedge v) \in k(u) \vee k(v)$, proving the claim. Since $\uparrow(u \vee v)$ is compact we have by the above remark that $k(u \vee v) \vee J = L$ and hence that $k(u) \vee k(v) \vee J = L$. Thus

$$(k(u) \vee J) \vee (k(v) \vee J) = L,\ (k(u) \vee J) \wedge (k(v) \vee J) = J \text{ since}$$

$k(u) \wedge k(v) = k(0) = 0$. Furthermore, $k(u) \vee J \neq J$ for otherwise $k(u) \subseteq J$ and hence $k(v) \vee J = L$. Since $J = \bigvee\{k(x) \in \alpha L \mid x \ll 1\}$ we have by the compactness of γL that $k(v) \vee k(x) = L$ for some $x \ll 1$. Taking joins we then have $v \vee x = 1$. Hence $\uparrow(v \vee x) = \uparrow 1$ iscompact so that $x \in N_2$. Now since $x \ll 1$ we have $\uparrow x^*$ is compact and therefore $x^* \in N_2$. Thus $0 = x \wedge x^* \in N_2$ implying that $\uparrow v$ is compact, a contradiction. Hence $k(u) \vee J \neq J$ and similarly $k(v) \vee J \neq J$. Thus $\uparrow J$ is disconnected. ∎

Theorem 7.3.2. *[3] Let L be regular continuous. Then every compactification $h\colon M \to L$ has a remainder which is compact and connected if and only if whenever $\uparrow(u \vee v)$ is compact and $u \wedge v = 0$ in L, then either $\uparrow u$ is compact or $\uparrow v$ is compact.*

Proof. We prove the sufficiency first. Let $h : M \to L$ be a compactification of L. To avoid unnecessary symbols let us simply denote the unique $a_L \in M$ determining the remainder of L in M described earlier by $a \in M$. Now $\uparrow a$ is compact, being a closed sublocale of compact M. Assume $\uparrow a$ is not connected. Then there exists $c, d \in \uparrow a,\ c, d \neq a$ such that $c \vee d = 1$ and $c \wedge d = a$. Since M, being compact regular, is normal there exists $f, g \in M$ such that $c \vee f = 1,\ d \vee g = 1$, and $f \wedge g = 0$. Now $(c \vee f) \wedge (d \vee g) = 1$ implies $((c \vee f) \wedge d) \vee ((c \vee f) \wedge g) = 1$ which implies $(c \wedge d) \vee (f \wedge d) \vee (c \wedge g) \vee (f \wedge g) = 1$ and thusimplies $a \vee f \vee g = 1$. Consider the frame $\downarrow a$. We claim that in this frame $\uparrow^{\downarrow a}(f \wedge a) \vee (g \wedge a)$ is compact. For this consider the map $\phi : \uparrow(f \vee g) \to \uparrow^{\downarrow a} a \wedge (f \vee g)$ given by $\phi(x) = x \wedge a$. We have $\phi(f \vee g) = a \wedge (f \vee g)$, $\phi(1) = 1 \wedge a = a$ so ϕ preserves top and bottom. It is then clearly a frame map. Furthermore, $\phi(x) = \phi(y)$ implies $x \wedge a = y \wedge a$ which implies$x = x \wedge (a \vee f \vee g) = (x \wedge a) \vee (x \wedge (f \vee g)) = (y \wedge a) \vee (f \vee g) \leq y \vee y = y$, so that$x = y$, by symmetry. Thus ϕ is one to one. Furthermore, ϕ is also onto. Indeed, take$y \in M,\ a \wedge (f \vee g) \leq y$ and $y \leq a$. Then

$\phi(y \vee (f \vee g)) = (y \vee (f \vee g)) \wedge a = (y \wedge a) \vee ((f \vee g) \wedge a) = y \vee ((f \vee g) \wedge a) = y.$

Thus $\uparrow(f \vee g) \cong \uparrow^{\downarrow a} a \wedge (f \vee g)$, and since $\uparrow f \vee g$ is compact, being a closed sublocale of M, $\uparrow^{\downarrow a} a \wedge (f \vee g)$ must also be compact. Thus $\uparrow^{\downarrow a}(f \wedge a) \vee (g \wedge a)$ is compact. Since $h\colon \downarrow a \to L$ is an isomorphism and $\uparrow^{\downarrow a}(f \wedge a) \vee (g \wedge a)$ is compact, we must have $\uparrow h(f) \vee h(g)$ compact in L. Since $h(f) \wedge h(g) = 0$, we must have either $\uparrow h(f)$ compact or $\uparrow h(g)$ compact, say $h(g)$ compact.

Now take any $0 \neq z \prec f$. Then $z^* \vee f = 1$ and hence $h(z^*) \vee h(f) = 1$. Now $h(z^*) = \bigvee \{w \in L \mid w \ll h(z^*)\}$, and hence $h(f) \vee \bigvee \{w \in L \mid w \ll h(z^*)\} = 1$, that is, $\bigvee \{h(f) \vee w \mid w \ll h(z^*)\} = 1$. Due to compactness of $\uparrow h(f)$ we can therefore find $w \ll h(z^*)$ such that $h(f) \vee w = 1$. Now $w \ll h(z^*)$ implies $w \ll 1$ and hence $r(w) \leq a$ by the definition of a. Now $h(f^*) \leq w$ since $h(f) \vee w = 1$ and hence $f^* \leq r(w) \leq a$. Thus since $g \wedge f = 0$, we have $g \leq f^* \leq a$, and therefore $1 = d \vee g \leq d \vee a = d$ since $a \leq d$. Hence $c = c \wedge 1 = c \wedge d = a$, a contradiction since $c \neq a$. Thus the remainder $\uparrow a$ is connected.

For necessity suppose every compactification $h : M \to L$ has a remainder which is compact and connected. Assume the condition on L is not satisfied. Then there exists $u, v \in L$, $\uparrow (u \vee v)$ compact but neither $\uparrow u$ nor $\uparrow v$ is compact. It follows that $u \neq 0$ and $v \neq 0$. From Section 3 in [3] we can construct a compactification of L such that the remainder of L in it is disconnected. Thus the condition on L must be satisfied. □

We end this this section with the following theorem which is also the main result in this section.

Theorem 7.3.3. *[3] The following conditions are equivalent for non-compact regular continuous frame L.*

1. *The least compactification of L is perfect.*

2. *Whenever $\uparrow(u \vee v)$ is compact, $u, v \in L$, $u \wedge v = 0$ then either $k(u) \vee J = L$ or $k(v) \vee J = L$ where $J\,L$ is the unique element in αL such that $\downarrow J \to L$ is an isomorphism, and $k : L \to \alpha L$ is the right adjoint of $\vee$.*

3. *Whenever $\uparrow(u \vee v)$ is compact, $u, v \in L$, $u \wedge v = 0$ then either $\uparrow u$ is compact or $\uparrow v$ is compact.*

4. *For every compactification $h : M \to L$ the remainder of L in it is compact and connected.*

Proof. (1) $\Rightarrow$ (2): Assume $\vee: \alpha L \rightarrow L$ is perfect. Take $u, v \in L$, $u \wedge v = 0$ with $\uparrow(u \vee v)$ compact. By Lemma 7.3.4, $k(u \vee v) \vee J = L$. Since $(\alpha L, \vee)$ is perfect, we then have $k(u) \vee k(v) \vee J = L$. Now, we cannot have both $k(u) \subseteq J$ and $k(v) \subseteq J$, otherwise $J = L$ which is not possible. Thus either $k(u)$ cJ J or $k(v)$ cJ J. Hence, by the remarks at the end of Section 1 in [3] we have either $k(u) \vee J = L$ or $k(v) \vee J = L$.

(2) $\Rightarrow$ (3): Suppose $\uparrow(u \vee v)$ is compact, $u \wedge v = 0$. Then either $k(u) \vee J = L$ or $k(v) \vee J = L$, and hence by Lemma 7.3.4, either $\uparrow u$ is compact or $\uparrow v$ is compact.

(3) $\Rightarrow$ (1): We recall from Proposition 7.1.3 that if $h : M \rightarrow L$ is a compactification of L with $r : L \rightarrow M$ the right adjoint of h, then $h : M \rightarrow L$ is perfect if and only if the following conditions are satisfied: x <J $(u \vee u^*)$, $x \leq u$ implies x <J u for all $x, u \in L$, where <J is the associated strong inclusion arising from $h: M \rightarrow L$. In the present case $\vee: \alpha L \rightarrow L$ with right adjoint $k: L \rightarrow \alpha L$, we have to show $x \blacktriangleleft (u \vee u^*)$, $x \leq u$ implies $x \blacktriangleleft u$.

Consider first the case when $\uparrow(u \vee u^*)$ is compact. Then either $\uparrow u$ is compact or $\uparrow u^*$ is compact. Now $x \blacktriangleleft u \vee u^*$ implies $x \prec u \vee u^*$, and $x \leq u$ implies $x \prec u$: for, there exists v such that $x \wedge v = 0$, $v \vee u \vee u^* = 1$. Thus $x \wedge (v \vee u^*) = (x \wedge v) \vee (x \wedge u^*) = 0$ and $v \vee u^* \vee u = 1$ with a separating element $v \vee u^*$. If $\uparrow u$ is compact then $x \blacktriangleleft u$. If on theother hand, $\uparrow u^*$ is compact, then $x \prec u$ implies $u^* \prec x^*$ from which it follows that $\uparrow x^*$ is compact. This implies $x \blacktriangleleft u$ as well.

Now consider the case where $\uparrow(u \vee u^*)$ is not compact. Take $x \blacktriangleleft u \vee u^*$, $x \leq u$. As before, $x \prec u$. Also either $\uparrow x^*$ is compact or $\uparrow(u \vee u^*)$ is compact. Since the latter is not possible, we have $\uparrow x^*$ compact. Hence $x \blacktriangleleft u$ and thus $(\alpha L, \vee)$ is a perfect compactification. $\square$

Bibliography

[1] P. Alexandroff and P. Urysohn, *Sur les espaces topologiques compacts*, Bull. Int. Acad. Pol. Sci. Lett. Ser. A, 1923, 5-8.

[2] D. Baboolal, *Perfect compactifications of frames*, Czechoslovak mathematical journal, 61.3 (2011), 845-861.

[3] D. Baboolal, *Conditions under which the least compactification of a regular continuous frame is perfect*, Czechoslovak mathematical journal, 62.2 (2012), 505-515.

[4] P. Bacon, *The compactness of countably compact spaces*, Pacific Journal of mathematics, 32.3 (1970), 587-592.

[5] R.N. Ball and J. Walters-Wayland, *C-and C^*-quotients in pointfree topology*, Dissertation Mathematicae (Rozprawy Mat.) 412, 2002.

[6] B. Banaschewski , *Compactification of frames*, Math.Nachr. 149.1 (1990), 105-115.

[7] B. Banaschewski, *Completion in pointfree topology*, in: SoCat94, in: Lecture Notes in Math. and Appl. Math., vol.2, Univ. of CapeTown, 1996.

[8] B. Banaschewski, *The real numbers in pointfree topology*, Vol. 12, Universidad de Coimbra, 1997.

[9] B. Banaschewski and S.S. Hong, *Filters and strict extensions of frames*, Kyungpook Math. J, 39(1999), 215-230.

[10] B. Banaschewski and S.S. Hong, *Extension by continuity in pointfree topology*, Appl.Categorical Structures, 8, 39(2000), 475-486.

[11] B. Banaschewski and S.S. Hong, *General filters and strict extensions in pointfree topology*, Kyungpook Math. J, 42(2002), 273-283.

[12] B. Banaschewski and S.S. Hong, *Variants of compactness in pointfree topology*, Kyungpook Math. J, 45(2005), 455-470.

[13] B. Banaschewski and C.J. Mulvey, *Stone-Čech compactification of locales I*, Houston J. Math. 6 (1980), 301312.

[14] B. Banaschewski and C.J. Mulvey, *Stone-Čech compactification of locales II*, Journal of pure and applied Algebra, 33.2 (1984), 107-122.

[15] R. Beazer and S. MacNab, *Modal extensions of Heyting algebras*, Colloq. Math. 41, (1979), 112.

[16] P. Bhattacharjee and I. Naidoo, *Another note on precompact uniform frames*, Topology and its Application, 275(2020).

[17] T. Dube,*An algebraic view of weaker forms of realcompactness*, algebra universalis, 55.2 (2006), 187-202.

[18] T. Dube, *Balanced and closed-generated filters in frames*, Quaestiones Mathematicae 26.1 (2003), 73-81.

[19] T. Dube, *On the ideal of functions with compact support in pointfree function rings*, Acta Mathematica Hungarica, 129.3 (2010), 205-226.

[20] T. Dube, *On compactness of frames*, algebra universalis, 51.4 (2004), 411-417.

[21] T. Dube, *Extending and contracting maximal ideals in the function rings of pointfree topology*, Bulletin mathmatique de la Socit des Sciences Mathmatiques de Roumanie, (2012), 365-374.

[22] T. Dube and J. Walters-Wayland, *Coz-onto frame maps and some applications*, Appl. Categor. Struct., 15 (2007), 119133.

[23] T. Dube, I. Naidoo and C.N. Ncube, *Isocompactness in the category of locales*, Applied Categorical Structures 22.5 (2014), 727-739.

[24] T. Dube and I. Naidoo, *More on uniform paracompactness in pointfree topology*, Math. Slovaca 65 (2015), 273288.

[25] A.A. Estaji, A.M. Darghadam, and H. Yousefpour, *On maximal ideals of* $\mathsf{R}^{\infty}L$, Journal of Algebraic Systems, 6.1(2018), 43-57.

[26] A.A. Estaji, A. Karimi Feizabadi and M. Abedi, *Strongly fixed ideals in* $C(L)$ *and compact frames*, Archivum Mathematicum 51.1 (2015), 1-12.

[27] A.K. Feizabadi and M.M. Ebrahimi, *Pointfree prime representation of real Riesz maps*, Algebra Universalis 54.3 (2005), 291-299.

[28] M. Fréchet, *Généralisation d'un théorème de Weierstrass*, Comptes Rendus Acad. Sci. Paris, 39 (1904), 848-850.

[29] D. Harris , *Extension closed and cluster closed subspaces*, Canadian Journal of Mathematics, 24.6 (1972), 1132-1136.

[30] S.S. Hong, *Convergence in frames*, Kyungpook Math. J, 35(1995), 85-91.

[31] S.S. Hong, *Simple extensions of frames*, Proc. Recent Devel. of Gen. Top. and its Appl., Math. Research, 67 (1992), 156-159.

[32] P.T. Johnstone, *Stone spaces*, Vol. 3, Cambridge university press, 1982.

[33] J. Paseka, *Conjunctivity in quantales*, Archivum Mathematicum, 24.4 (1988), 173-179.

[34] J. Paseka and B. Šmarda, *T_2-frames and almost compact frames*, Czech.Math.J., 42(1992),385-402.

[35] A. Picado and A. Pultr, *Frames and Locales*: *Topology without points*, Frontiers in mathematics, Springer, Basel, (2012).

[36] M. Sakai, *On cl-isocompactness and weak Borel completeness*, Tsukuba J. Math. 8(2), (1984), 377382.

[37] H. Simmons, *A framework for topology*, Studies in Logic and the Foundations of Mathematics, Vol. 96. Elsevier, (1978), 239-251.

[38] E.G. Sklyarenko, *Some questions in the theory of bicompactifications*, Am. Math. Soc. Transl. 58 (1966), 216-244.

[39] M. Stanojevi, *Locally finite hyperspace topology of isocompact spaces*, Filomat (1997), 49-54.