A TYPE OF 'INVERSENESS' OF CERTAIN DISTRIBUTIONS AND THE INVERSE NORMAL DISTRIBUTION

by

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I declare that the essay for the degree Master of Science at the University of the North hereby submitted by me, had not previously been submitted by me for a degree at this or at another university, and that it is my own work in design and in execution and that all material contained herein is recognized.

To the memory of my late father.

PREFACE

This essay accounts for a fifth of the requirements for the masters degree. Four papers have already been written to this end. Originality of research is not claimed, and the essay is to be regarded as the equivalent of an examination paper, and as a self-preparation for doctorate research.

Part I of the essay deals with what the writer calls a form of 'inverseness' of certain distributions. Common univariate distributions are treated, and multivariate extensions derived. Limiting forms of 'inverseness' are also given.

In Part II greater attention is given to the inverse normal distribution. Its properties and characteristics are studied. Estimation of parameters and tests of hypotheses concerning them receive attention. Reference is repeatedly made to Wald's evaluation of a certain Fourier transform:

$$\frac{1}{2\Pi}\int_{-\infty}^{\infty} e^{\frac{a\mu}{\sigma^2}-\frac{a}{\sigma^2}\sqrt{\mu^2-2\sigma^2 iu}\ -iun}\,du$$

$$= \frac{ae^{\frac{a\mu}{\sigma^2}}}{\sqrt{2\Pi\sigma^2}} e^{-\frac{a^2}{2\sigma^2 n}-\frac{\mu^2 n}{2\sigma^2}} n^{-1\frac{1}{2}} .$$

A short account is given of how the inverse normal distribution can be regarded as a sampling distribution in a renewal process when the sampling is from a normal parent.

The last portion of the essay gives a brief account of distributions related to the inverse normal distribution, and of the distributions of the sample arithmetic and harmonic means.

I am indebted to my tutor and supervisor, Prof. P.C.D. Oliver, for suggesting this study, and for valuable advice, contributions and comments in the preparation of this essay.

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CONTENTS

PART I

Atype of 'inverseness' of certain distributions

Chapter 1

COMMON DISTRIBUTIONS

1.1 The inverse Gaussian distributions of Tweedie and that of Wald

In papers by Tweedie [1] and Schrödinger (see Moran [2]), the name 'inverse Gaussian distributions' referred to a family of continuous distributions of a random variable X>0 whose density function has any one of the forms

(i) $$f_x(x;\alpha,\lambda) = \frac{\lambda^{\frac{1}{2}}}{(2\pi)^{\frac{1}{2}}} e^{-\alpha\lambda x+\lambda(2\alpha)^{\frac{1}{2}}-\lambda/2x} x^{-3/2}$$

(ii) $$f_x(x;\mu,\lambda) = \frac{\lambda^{\frac{1}{2}}}{\sqrt{2\pi}} e^{-\frac{\lambda(x-\mu)^2}{2\mu^2 x}} x^{-3/2}$$

(iii) $$f_x(x;\mu,\phi) = \sqrt{\frac{\mu\phi}{2\pi}} e^{-\frac{\phi x}{2\mu}+\phi-\frac{\mu\phi}{2x}} x^{-3/2}$$

(iv) $$f_x(x;\phi,\lambda) = \sqrt{\frac{\lambda}{2\pi}} e^{-\frac{\phi^2 x}{2\lambda}+\phi-\frac{\lambda}{2x}} x^{-3/2}$$

Where the parameters α, λ and ϕ are real and positive. It was found that certain properties of these distributions were exhibited in the study of Brownian movement. The same type of distribution is derived in connection with a special case of

Wald's....

sequential likelihood ratio test. In Wald's sense it refers to the approximate distribution of N, the number of observations needed to terminate the process of sampling. It is 'inverse' in the sense that it is the number of observations that is of interest, not the outcome as such. The two cases above will be referred to in more detail in Part II of this essay. In the present discussion, the word 'inverse' will be used in a broader sense and will have meanings like 'negative', 'ratio', 'reversal of normal order of procedure' and 'complementary events' as we illustrate in the sequel.

1.2 Univariate distributions

For the sake of reference, we repeat the well-known 'inverse' relationships that exist between some common standard distributions.

The beta and binomial distributions

If X has beta distribution with parameters n and k and if Y has binomial distribution with parameters m and p, we know that

$$\int_{o}^{p} \frac{n!}{(k-1)!(n-k)!} y^{k-1}(1-y)^{n-k}dy = \sum_{x=k}^{n} \frac{n!}{x!(n-x)!} p^{x}(1-p)^{n-x}$$

i.e. $P(Y<p) = 1-P(X<k)$

or equivalently

$$F_y(p)+F_X(k-1) = 1 \qquad \ldots\ldots \qquad (1.1)$$

From (1.1) we see that the events $\{Y<p\}$ and $\{X\leq k-1\}$ are complements of one another. In words (1.1) would mean: the ratio of the number of 'successes' out of n trials is less than a number p if the number of 'failures' is more than k.

The Poission and gamma distributions

For a variate from Poisson distribution with parameter λ, it is easily seen that

$$\sum_{k=o}^{x} \frac{e^{-\lambda}\lambda^{k}}{k!} = \frac{1}{\Gamma(x+1)} \int_{\lambda}^{\infty} e^{-y} y^{(x+1)-1} dy$$

i.e. $P(X\leq x) = P(Y>\lambda)$ or $P(X\leq x) = 1-P(Y\leq\lambda)$

i.e. $F_X(x)+F_y(\lambda) = 1. \qquad \ldots\ldots \qquad (1.2)$

The 'inverseness' is as in the binomial-beta case above. (1.2) is justified since we know that in a homogenuous Poisson process the number of events X is a Poisson variate whilst the waiting time Y up to the x-th event has gamma distribution. In other words instead of counting the number of events in a fixed length of time, we interest ourselves in the length of time when the x-th event occurs i.e. there is a reversal of rôles between X and Y.

<u>The binomial and the negative binomial distributions</u>.

We know that

$$P(X<r) = 1-P(Y\leq n-r)$$

$$\text{i.e. } F_X(r-1)+F_Y(n-r) = 1 \qquad \ldots\ldots \qquad (1.3)$$

where X has binomial distribution with parameters n and p, and Y has negative-binomial distribution with parameters r and p. Here the 'inverseness' is that instead of observing the number of 'successes' X in n trials, we may do the opposite and concern ourselves with the number of trials needed until exactly x 'successes' are obtained. The prefix 'negative' is therefore synonymous to 'inverse'. Infact some authors (e.g. Wilks) term the variate Y as having binomial-waiting time distribution.

The beta and the negative-hypergeometric distributions.

An analogous relation could not be found in this instance. However, let N have negative-hypergeometric distribution with parameters, M, p and x, then

$$P(N=n) = \frac{\binom{n-1}{x-1}\binom{M-n}{Mp-x}}{\binom{M}{Mp}}$$

$$= \frac{\frac{x}{n}\binom{Mp}{x}\binom{M-Mp}{n-x}}{\binom{M}{n}}$$

$$= \frac{\frac{x}{n}\binom{M-Mp}{n-x}}{B(x+1,Mp-x+1)} \int_0^1 z^{n-1}(1-z)^{M-n}dz \qquad \ldots\ldots \quad (1.4)$$

Dieulefait [4] has shown, using (1.4), that the random variable $Y = \frac{N-x}{M-Mp}$ has a limiting beta $(x,Mp-x+1)$ distribution in the following manner: The characteristic function of N is derived, and is found to be

$$\phi_N(\theta) = E(e^{\theta N}) = \frac{1}{B(x,Mp-x+1)} e^{x\theta} \int_0^1 z^{x-1}(1-z)^{Mp-x}(1-z+ze^{\theta})^{M-Mp} dz$$

...... (1.5)

where θ = it.

The limiting distribution of Y is investigated by considering its characteristic function i.e.

$$\phi_y(\theta) = e^{\frac{x}{M-Mp}} E\left[e^{\frac{N}{M-Mp}\theta}\right]$$

$$= \frac{1}{B(x,Mp-x+1)} \int_0^1 z^{x-1}(1-z)^{Mp-x}\left[1-z+ze^{\frac{\theta}{M-Mp}}\right]^{M-Mp} dz$$

$$= \frac{1}{B(x,Mp-x+1)} \int_0^1 z^{x-1}(1-z)^{Mp-x}\left(1-z+z\left[1+\frac{\theta}{M-Mp}+\frac{\psi(\theta)}{M-Mp}\right]\right)^{M-Mp} dz$$

(where $\frac{\psi(\theta)}{M-Mp} \to 0$ as $(M-Mp) \to \infty$)

Thus on taking the limit as $(M-Mp) \to \infty$, we have

$$\lim_{(M-Mp)\to\infty} \phi_y(\theta) = \frac{1}{B(x,Mp-x+1)} \int_0^1 z^{x-1}(1-z)^{Mp-x} e^{\theta z} dz$$

...... (1.6)

But the right hand side of (1.6) defines the characteristic function of a beta variate. This establishes a (limiting) relationship between the hypergeometric and the beta distributions. Notice that it is not the limiting equivalent of the relation between binomial and beta distributions. Nor is it the same as the limiting distribution of the proportion of 'successes', $\frac{x}{n}$, in the n binomial trials.

$$\phi_{\frac{X}{n}}(t) = E(e^{X\frac{it}{n}}) = (pe^{\frac{it}{n}}+q)^n$$

$$= (pe^{\frac{it}{n}}+1-p)$$

$$= (1+p[e^{\frac{it}{n}}-1])^n$$

$$= (1+p[\frac{it}{n}+\frac{(it)^2}{2!n^2}+\ldots])^n$$

so $\lim_{n\to\infty} \phi_{\frac{X}{n}}(t) = e^{pit} = \phi_p(t)$

which is a useless result since p is a constant. Inverting the rôles of 'successes' and 'failures', it follows that $\frac{X}{Mp}$ in the long-run has a beta (Mp-x+1,x) distribution.

1.3 A multivariate extension of the binomial-beta relationship

Consider the random variables $X_1,\ldots X_k$ from a multinomial distribution with parameters $n,p_1,\ldots,p_{k-1},p_k=1-p_1-\ldots-p_{k-1}$ and $x_1+x_2+\ldots+x_k=n$. For k=2, then

$$P(X_1\geq i,X_2\geq j) = \sum_{x_2=j}^{n-i}\sum_{x_1=i}^{n-x_2}\frac{n!}{x_1!x_2!(n-x_1-x_2)!}p_1^{x_1}p_2^{x_2}(1-p_1-p_2)^{n-x_1-x_2}$$

$$= \sum_{x_2=j}^{n-i}\frac{n!p_2^{x_2}}{x_2!}\sum_{x_1=i}^{n-x_2}\frac{1}{x_1!(n-x_1-x_2)!}p_1^{x_1}(1-p_1-p_2)^{n-x_1-x_2}$$

$$= \sum_{x_2=j}^{n-i}\frac{n!p_2^{x_2}(1-p_2)^{n-x_2}}{x_2!(n-x_2)!}$$

$$\sum_{x_1=i}^{n-x_2}\frac{(n-x_2)!}{x_1!(n-x_1-x_2)!}\left(\frac{p_1}{1-p_2}\right)^{x_1}\left(1-\frac{p_1}{1-p_2}\right)^{n-x_1-x_2}$$

$$= \sum_{x_2=j}^{n-i}\frac{n!p_2^{x_2}(1-p_2)^{n-x_2}}{x_2!(n-x_2)!}\frac{1}{B(i,n-x_2-i+1)}$$

$$\int_0^{\frac{p_1}{1-p_2}} u^{i-1}(1-u)^{n-x_2-i}du$$

$$= \sum_{x_2=j}^{n-i} \frac{n!p_2^{x_2}}{x_2!} \frac{1}{(i-1)!(n-x\ -i)!} \int_0^{p_1} v^{i-1}(1-p_2-v)^{n-x_2-i}dv$$

(if we put $u = \frac{v}{1-P_2}$)

$$= \frac{n!}{(i-1)!(n-i)!} \int_0^{p_1} v^{i-1} \sum_{x_2=j}^{n-i} \binom{n-i}{x_2} p_2^{x_2}(1-p_2-v)^{n-i-x_2}dv$$

$$= \frac{n!}{(i-1)!\,(n-i)!} \int_0^{p_1} v^{i-1}(1-v)^{n-i} \sum_{x_2=j}^{n-i} \binom{n-i}{x_2} \left(\frac{p_2}{1-v}\right)^{x_2}$$

$$\left(1-\frac{p_2}{1-v}\right)^{n-i-x_2} dv$$

$$= \frac{n!}{(i-1)!(n-i)!} \int_0^{p_1} v^{i-1}(1-v)^{n-i} \frac{1}{B(j,n-i-j+1)}$$

$$\int_0^{\frac{p_2}{1-v}} w^{j-1}(1-w)^{n-j-j}dwdv$$

$$= \frac{n!}{(i-1)!(j-1)!(n-i-j)!} \int_0^{p_1}\int_0^{p_2} v^{i-1}u^{j-1}(1-v-u)^{n-i-j}dudv$$

(with $w = \frac{u}{1-v}$)

$$= \frac{\Gamma(n+1)}{\Gamma(i)\Gamma(j)\Gamma(n-i-j+1)} \int_0^{p_1}\int_0^{p_2} v^{i-1}u^{j-1}(1-v-u)^{n-i-j}\,du\,dv$$

i.e. $P(X_1 \geqslant i, X_2 \geqslant j) = P(U \leqslant p_1, V \leqslant p_2)$

where U and V have a joint Dirichlet distribution (one of the multivariate extensions of the beta distribution) with parameters n, i and j. We shall now prove by induction the k-variate extension.

Suppose for k=r

$$\frac{n!}{(n-\sum_{j=1}^{r} i_j)!\prod_{j=1}^{r}(i_j-1)!}\int_0^{p_1}\cdots\int_0^{p_r}\left(\prod_{j=1}^{r} a_j^{i_j-1}\right)\left(1-\sum_{j=1}^{r} a_j\right)^{n-\sum_{j=1}^{r} a_j}\prod_{j=1}^{r} da_j$$

$$= \sum_{x_r}\cdots\sum_{x_1}\frac{n!}{(n-\sum_{j=1}^{r} x_j)!\prod_{j=1}^{r} x_j!}\left(\prod_{j=1}^{r} p_j^{x_j}\right)\left(1-\sum_{j=1}^{r} p_j\right)^{n-\sum_{j=1}^{r} a_j}$$

Now for k=r+1, we have

$$\sum_{x_{r+1}}\cdots\sum_{x_1}\frac{n!}{(n-\sum_{j=1}^{r+1} x_j)!(\prod_{j=1}^{r+1} x_j!)}\left(\prod_{j=1}^{r+1} p_j^{x_j}\right)\left(1-\sum_{j=1}^{r+1} p_j\right)^{n-\sum_{j=1}^{r+1} x_j}$$

$$= \sum_{x_{r+1}} \frac{n!\,p_{r+1}^{x_{r+1}}}{x_{r+1}!} \sum_{x_r} \cdots \sum_{x_1} \frac{\left(\prod_{j=1}^{r} p_j^{x_j}\right)\left(1-\sum_{j=1}^{r+1} p_j\right)^{[n-x_{r+1}]-\sum_{j=1}^{r} x_j}}{\left(n-\sum_{j=1}^{r+1} xj\right)!\left(\prod_{j=1}^{r} xj!\right)}$$

$$= \sum_{x_{r+1}} \frac{n!\,p_{r+1}^{x_{r+1}}}{(n-x_{r+1})!\,x_{r+1}!} \sum_{x_r} \cdots \sum_{x_1} \frac{(n-x_{r+1})!}{\left(n-x_{r+1}-\sum_{j=1}^{r} x_j\right)!\left(\prod_{j=1}^{r} x_j!\right)} \left(\prod_{j=1}^{r} p_j^{x_j}\right)$$

$$\left(1-p_{r+1}-\sum_{j=1}^{r} p_j\right)^{n-x_{r+1}-\sum_{j=1}^{r} x_j}$$

$$= \sum_{x_{r+1}} \frac{n!\,p_{r+1}^{x_{r+1}}(1-p_{r+1})^{n-x_{r+1}}}{x_{r+1}!\,(n-x_{r+1})!} \sum_{x_r} \cdots \sum_{x_1} \frac{(n-x_{r+1})!}{\left(n-x_{r+1}-\sum_{j=1}^{r} x_j\right)!\left(\prod_{j=1}^{r} x_j!\right)}$$

$$\left(\prod_{j=1}^{r}\left[\frac{p_j}{1-p_{r+1}}\right]^{x_j}\right)\left(1-\sum_{j=1}^{r}\frac{p_j}{1-p_{r+1}}\right)^{n-x_{r+1}-\sum_{j=1}^{r} x_j}$$

$$= \sum_{x_{r+1}} \frac{n!\,p_{r+1}^{x_{r+1}}(1-p_{r+1})^{n-x_{r+1}}}{x_{r+1}!\,(n-x_{r+1})!} \frac{(n-x_{r+1})!}{\left[\prod_{j=1}^{r}(i_j-1)!\right]\left(n-x_{r+1}-\sum_{j=1}^{r} i_j\right)!}$$

$$\times \int_{o}^{\frac{p_1}{1-p_{r+1}}} \cdots \int_{o}^{\frac{p_r}{1-p_{r+1}}} \left(\prod_{j=1}^{r} a_j^{i_j-1}\right)\left(1-\sum_{j=1}^{r} a_j\right)^{n-x_{r+1}-\sum_{j=1}^{r} i_j} \left(\prod_{j=1}^{r} da_j\right)$$

(from assumption)

$$= \sum_{x_{r+1}} \frac{n!\,p_{r+1}^{x_{r+1}}}{x_{r+1}!\left(\prod_{j=1}^{r}(i_j-1)!\right)\left(n-x_{r+1}-\sum_{j=1}^{r} i_j\right)!} \int_{o}^{p_1} \cdots \int_{o}^{p_r} \left(\prod_{j=1}^{r} b_j^{i_j-1}\right)$$

$$\times \left(1-p_{r+1}-\sum_{j=1}^{r} b_j\right)^{n-x_{r+1}-\sum_{j=1}^{r} i_j} \left(\prod_{j=1}^{r} db_j\right)$$

(with $a_j = \frac{b_j}{1-p_{r+1}}$)

$$= \frac{n!}{\left(\prod_{j=1}^{r}(i_j-1)!\right)} \int_0^{p_1}\cdots\int_0^{p_r} \left(\prod_{j=1}^{r} b_j^{i_j-1}\right) \sum_{x_{r+1}} \frac{p_{r+1}^{x_{r+1}}\left(1-p_{r+1}-\sum_{j=1}^{r} b_j\right)^{n-x_{r+1}-\sum_{j=1}^{r} i_j}}{x_{r+1}!\left(n-\sum_{j=1}^{r} i_j - x_{r+1}\right)!}$$

$$\prod_{j=1}^{r} db_j$$

$$= \frac{n!}{\left(\prod_{j=1}^{r}(i_j-1)!\right)} \int_0^{p_1}\cdots\int_0^{p_r} \left(\prod_{j=1}^{r} b_j^{i_j-1}\right) \frac{\left(1-\sum_{j=1}^{r} b_j\right)^{n-\sum_{j=1}^{r} i_j}}{\left(n-\sum_{j=1}^{r} i_j\right)!}$$

$$x \sum_{x_{r+1}} \binom{n-\sum_{j=1}^{r} i_j}{x_{r+1}} \left(\frac{p_{r+1}}{1-\sum_{j=1}^{r} b_j}\right)^{x_{r+1}} \left(1-\frac{p_{r+1}}{1-\sum_{j=1}^{r} b_j}\right)^{n-x_{r+1}-\sum_{j=1}^{r} i_j} \left(\prod_{j=1}^{r} db_j\right)$$

$$= \frac{n!}{\prod_{j=1}^{r}(i_j-1)!} \int_0^{p_1}\cdots\int_0^{p_r} \left(\prod_{j=1}^{r} b_j^{i_j-1}\right) \frac{\left(1-\sum_{j=1}^{r} b_j\right)^{n-\sum_{j=1}^{r} i_j}}{\left(n-\sum_{j=1}^{r} i_j\right)!} \frac{1}{B\left(i_{r+1}, n-\sum_{j=1}^{r+1} i_j - 1\right)}$$

$$\int_0^{\frac{p_{r+1}}{1-\sum_{j=1}^{r} b_j}} a_{r+1}^{i_{r+1}-1} \left(1-a_{r+1}\right)^{n-\sum_{j=1}^{r+1} i_j} da_{r+1} \left(\prod_{j=1}^{r} db_j\right)$$

(from assumption)

$$= \frac{n!}{\left(\prod_{j=1}^{r+1}(i_j-1)!\right)\left(n-\sum_{j=1}^{r+1}i_j\right)!}\int_o^{p_1}\cdots\int_o^{p_r}\int_o^{p_{r+1}}\left(\prod_{j=1}^{r+1}b_j^{i_j-1}\right)$$

$$\left(1-\sum_{j=1}^{r+1}b_j\right)^{n-\sum_{j=1}^{r+1}i_j}\left(\prod_{j=1}^{r+1}db_j\right)$$

(where $a_{r+1}=\dfrac{b_{r+1}}{1-\sum_{j=1}^{r+1}b_j}$) .

1.4 Poisson approximation to the multinomial distribution

It is known that for the univariate case, the Poisson distribution serves as a good approximation to the binomial, i.e.

$$\binom{n}{x}p^x(1-p)^{n-x} = \frac{e^{-\lambda}\lambda^x}{x!} \text{ for n large and fixed } \lambda = np.$$

The extension of the above to the k-dimensional case is: if the random variables $X_1,\ldots,X_k$ have multinomial distribution with parameters $n,p_1,\ldots,p_k$, i.e. if

$$f(x_1,\ldots,x_k) = \frac{n!}{(\Pi x!)(n-\Sigma x)!}(\Pi p^x)(1-\Sigma p)^{n-\Sigma x}$$

then
$$\lim_{\substack{n\to\infty \\ np_i=\lambda_i}} f(x_1,\ldots,x_k) = \lim_{\substack{n\to\infty \\ np_i=\lambda_i}} \frac{n^{(\Sigma x)}}{n^{\Sigma x}(\Pi x!)}\left(\Pi [np]^x\right)[1-\frac{\Sigma np}{n}]^{n-\Sigma x}$$

(where $n^{(\Sigma x)}$ is the Σx-th factorial power of n)

$$= \frac{(\Pi\lambda^x)e^{-\Sigma\lambda}}{(\Pi x!)}$$

Thus for n large the multinomial distribution approximates the product of *independent* Poisson variates. We give a numerical illustration. For simplicity we treat the case of two variates X and Y, i.e.

$$f_{x,y}(x,y) = \frac{n!}{x!y![n-x-y]!}p^x q^y(1-p-q)^{n-x-y}$$

Thus
$$\lim_{\substack{n\to\infty \\ np=\lambda \\ nq=\mu}} f_{x,y}(x,y) = \frac{e^{-\lambda}\lambda^x}{x!} \cdot \frac{e^{-\mu}\mu^y}{y!}$$

For n=30, p=0,06, q=0,04, 1-p-q=0,9 and λ=np=1,8, μ=nq=1,2

	Exact Mult.Prob^{ty}	Poisson Approx.	Percentage Error
(0,0)	0,042391	0,049787	17,4
(0,1)	0,084782	0,089616	5,7
(0,2)	0,036424	0,035846	1,58
(1,0)	0,084782	0,089616	5,7
(1,1)	0,109274	0,107540	1,5
(1,2)	0,067993	0,064524	5,1
(1,3)	0,027197	0,025809	5,1
(1,4)	0,007857	0,007427	5,4
(2,0)	0,081956	0,080650	1,6
(2,1)	0,101989	0,096786	5,1
(2,2)	0,061193	0,058071	5,1
(2,3)	0,023571	0,023228	1,5
(2,4)	0,006547	0,006968	6,4
(3,1)	0,061193	0,058071	5,1
(3,2)	0,035356	0,034842	1,4
(3,3)	0,013095	0,013937	6,4
(3,4)	0,003492	0,004181	19,4
(4,1)	0,026517	0,026132	1,4
(4,2)	0,014731	0,015679	6,4
(4,3)	0,005238	0,0062717	19,7
(5,1)	0,008839	0,009407	6,4
(5,2)	0,004714	0,005645	19,7
(5,3)	0,001606	0,002257	40,5
(5,4)	0,009816	0,000677	93
(6,5)	0,000271	0,000048	82

For n=100, p=0,01, q=0,03, 1-p-1=0,96, λ=np=1, μ=nq=3

(X,Y)	Exact Mult. Prob^{ty}	Poisson Approx.	Percentage Error
(o,0)	0,016870	0,018315	8,56
(1,1)	0,054367	0,054946	1,06
(1,2)	0,083249	0,082420	0,99
(1,3)	0,081117	0,082420	1,6
(2,1)	0,026639	0,027473	3,1
(2,2)	0,042058	0,041210	2
(2,3)	0,042058	0,041210	2,01
(3,1)	0,0093463	0,009157	2
(3,2)	0,014019	0,013736	2
(4,1)	0,002336	0,002289	3,7
(4,2)	0,00346836	0,0034341	0,99
(5,1)	0,00046244	0,00045789	0,99
(5,2)	0,00067922	0,000686836	1,1
(5,3)	0,00065799	0,000686836	4

A very clear pattern does not emerge, but it is clear that if X and Y do not depart much from their means, the approximation is fairly good; also that the larger n the smaller p and q the better the approximation.

1.5 The gamma approximation to the beta distribution (Wilks [5],)

For the univariate case, let U be a random variable with p.d.f

$$f_U(u) = \frac{\Gamma(n+1)}{\Gamma(k)\Gamma(n-k+1)} u^{k-1}(1-u)^{n-k}, o<u<1.$$

let t=nu then $|\frac{du}{dt}|=\frac{1}{n}$

so $$f_T(t) = \frac{\Gamma(n+1)}{\Gamma(k)\Gamma(n-k+1)}\left(\frac{t}{n}\right)^{k-1}\left(1-\frac{t}{n}\right)^{n-k}$$

which tends to $\frac{1}{\Gamma(k)} e^{-t} t^{k-1}$ as $n\to\infty$.

Further

$$F_{U_1,U_2}(t_1,t_2) = \frac{\Gamma(n+1)}{\Gamma(k_1)\Gamma(k_2)\Gamma(n-k_1-k_2+1)} \int_o^{t_1}\int_o^{t_2} u_1^{k_1-1}(u_2-u_1)^{k_2-1}$$

$$(1-u_2)^{n-k_2-k_1} du_1 du_2$$

and if $v_1=nu_1$ and $v_2=nu_2$, then

$$\lim_{n\to\infty} F_{V_1,V_2}(s_1,s_2) = \frac{1}{\Gamma(k_1)\Gamma(k_2)} \int_o^{s_1}\int_o^{s_2} v_1^{k_1-1}(v_2-v_1)^{k_2-1} e^{-v_2} dv_1 dv_2$$

For the multivariate case, let the random variables $v_1,\ldots,v_s$ have p.d.f

$$f_{v_1,\ldots,v_s}(v_1,\ldots,v_s) = \frac{(n+1)}{\Gamma(k_1)\ldots\Gamma(k_s)\Gamma(n-k_1-\ldots-k_s+1)} v_1^{k_1-1}\ldots v_s^{k_s-1}$$

$$\cdot(1-v_1-\ldots-v_s)^{n-k_1-\ldots k_s}$$

If we put $w_1=nv_1,\ldots,w_s=nv_s$ then

$$\left|\frac{\partial(v_1\ldots v_s)}{\partial(w_1,\ldots,w_s}\right| = \frac{1}{n^s} \quad , \text{ so}$$

$$f_{w_1,\ldots,w_s}(w_1,\ldots,w_s) = \frac{\Gamma(n+1)}{n^{k_1+\ldots+k_s}\Gamma(k_1)\ldots\Gamma(k_s)\Gamma(n-k_1\ldots-k_s+1)}$$

$$\cdot w_1^{k_1-1}\ldots w_s^{k_s-1}\left[1-\frac{w_1+\ldots+w_s}{n}\right]^{n-k_1\ldots k_s}$$

which tends to

$$\frac{1}{\Gamma(k_1)\ldots\Gamma(k_s)} w_1^{k_1-1}\ldots w_s^{k_s-1} e^{-\sum_{1=1}^{s} w_i}$$

i.e. the multivariate beta tends to the product of independent gamma variates. We have shown the 'inverse' of the multinomial is a multivariate beta, and that of the Poisson is the gamma. Also, we have shown that the multinomial can be approximated by a product of independent Poisson variates. We see therefore that there are similar correspondences between the distributions on the one hand and their 'inverses' on the other, i.e. a form of duality.

Multinomial	———	Dirichlet (beta)
|		|
Independent Poissons	———	Independent gammas

PART II

The inverse normal distribution

Chapter 2

GENESIS, DEFINITION AND CHARACTERISTICS

2.1 The Brownian movement approach

Schrödinger (see Moran [2]), Richards [6], and Cox and Miller [7]) give the derivation of the inverse normal distribution in the study of *Brownian motion*. Consider the Wiener process $\{X(t), X(o)=0, t \geqslant 0\}$ with covariance kernel $\sigma^2 t$ and mean value μt and an absorbing barrier x=a. The transition probability function of X(t) satisfies the diffusion equation

$$\frac{1}{2}\sigma^2 \frac{\partial^2 p(x;t)}{\partial x^2} - \mu \frac{\partial p(x;t)}{\partial x} = \frac{\partial p(x;t)}{\partial t} \, (x<a) \qquad \ldots\ldots (2.1)$$

(Kolmogorov forward equation),

subject to the conditions

$$p(x;o) = \delta(x)$$

$$p(a;t) = 0 \ , \ t>o$$

where p(x;t) is the probability that the particle will be at x in time t. Denote by $p(x_o, x; t)$ the probability that the particle will move to x in time t when it was initially at state x_o. If the process was initially at $x=x_o$, then

$$p_1(x_o,x;t) = \frac{1}{\sqrt{2\Pi\sigma^2 t}} e^{-\frac{(x-x_o-\mu t)^2}{2\sigma^2 t}} \quad \text{is}$$

a general solution of (2.1). When $X(o)=x_o=0$, this reduces to

$$p_1(o,x;t) = \frac{1}{\sigma\sqrt{2\Pi}} e^{-\frac{(x-\mu t)^2}{2\sigma^2 t}} \quad .$$

Further the linear combination

$p(x,t) = p_1(o,x;t) + Ap_1(x_o,x;t)$ also satisfies the diffusion equation, where A is chosen such that the boundary conditions are satisfied. Using the method of images, the absorbing barrier x=a is regarded as the 'image' of the initial state, hence x_o is put equal to 2a. If A has the value $A = -\exp \frac{2\mu a}{\sigma^2}$, then the solution is

$$p(x,t) = \frac{1}{\sqrt{2\Pi\sigma^2 t}} \quad e^{-\frac{(x-\mu t)^2}{2\sigma^2}} - e^{\frac{2\mu a}{\sigma^2} - \frac{(x-2a-\mu t)^2}{2\sigma^2 t}}$$

Since $P(X(t)<a|X(o)=0) = P(T>t)$

(where T is the time to absorption at x=a), i.e. $$\int_{-\infty}^{a} p(x,t)\,dx = P(T>t),$$

we have that the p.d.f of T, the first passage time, is

$$f_T(t) = -\frac{t}{dt}\int_{-\infty}^{a} p(x,t)\,dx$$

$$= -\frac{t}{dt}\left[\frac{1}{\sqrt{2\Pi\sigma^2 t}}\int_{-\infty}^{a} e^{-\frac{(x-\mu t)^2}{2\sigma^2 t}}\,dx\right.$$

$$\left. - \frac{e^{\frac{2\mu a}{\sigma^2}}}{\sigma\sqrt{2\Pi}}\int_{-\infty}^{a} e^{-\frac{(x-2a-\mu t)^2}{2\sigma^2 t}}\,dx\right]$$

$$= -\frac{d}{dt}\left[\Phi\left(\frac{a-\mu t}{\sigma\sqrt{t}}\right) - e^{\frac{2\mu a}{\sigma^2}}\Phi\left(\frac{-a-\mu t}{\sigma\sqrt{t}}\right)\right]$$

where $$\Phi(x) = \frac{1}{\sqrt{2\Pi}}\int_{-\infty}^{x} e^{-\frac{y^2}{2}}\,dy$$

Since $$\frac{d}{dx}\int_{\phi(x)}^{\psi(x)} g(t,x)\,dt = \psi'(x)g(\psi(x),x) - \phi'(x)g(\phi(x),x)$$

$$+ \int_{\phi(x)}^{\psi(x)} \frac{\partial}{\partial x} g(t,x)dt,$$

we have

$$f_T(t) = \frac{-\mu\sigma\sqrt{t} - \frac{1}{2}a\sigma t^{-\frac{1}{2}} + \frac{\mu\sigma}{2}}{\sigma t} e^{-\frac{(a-\mu t)^2}{2\sigma^2 t}}$$

$$- e^{\frac{2a\mu}{\sigma^2}} \frac{-\mu\sigma\sqrt{t} + \frac{a\sigma t^{\frac{1}{2}}}{2} + \frac{\mu\sigma}{2}}{\sigma t} e^{-\frac{(a+\mu t)^2}{2\sigma^2 t}}$$

$$= \frac{a}{\sigma\sqrt{2\Pi t^3}} e^{-\frac{(a-\mu t)^2}{2\sigma^2 t}}, \quad t \geq 0.$$

2.2 The sampling inspection or sample size approach.

Wald ([3a] or [3b]), in connection with his sequential analysis, deals with the *sample size* N as a *variable*, rejecting the null hypothesis when $\sum_{i=1}^{N} z_i \geq \ln A = a$ and accepting when $\sum_{i=1}^{N} z_i \leq \ln B = b$ where $z_i, i=1,2,\ldots,N$, is the i-th observation.

The identity $E(e^{Z_N t}) = E(e^{Z_n t + [Z_N - Z_n]t}) = [\phi(t)]^N$,

where $\phi(t)=E(e^{tz})$ and $Z_i = z_1 + \ldots + z_i$, the z_i being independently distributed, is used to arrive at what Wald calls his fundamental identity,

$$E(e^{z_n t}[\phi(t)]^n) = 1.$$

Further, it is assumed that $-\log\phi(t) = \tau$ where τ is an imaginary quantity, and that the equation $-\log\phi(t) = 0$ has only two real roots t_1 and t_2. Now if the z_i are independent $n(\mu,\sigma^2)$, then

$-\log\phi(t) = -\mu t - \frac{\sigma^2 t^2}{2} = \tau$, and on solving for t, we have that

$$t_1 = -\frac{\mu}{\sigma^2} - \frac{1}{\sigma^2}\sqrt{\mu^2 - \sigma^2\tau}$$

$$t_2 = -\frac{\mu}{\sigma^2} + \frac{1}{\sigma^2}\sqrt{\mu^2 - \sigma^2\tau} \quad .$$

In this case the fundamental identity becomes

$$P(Z_n \leq \log B) B^t E(e^{\tau N} | Z_n \leq \log B)$$

$$+ P(Z_n \geqslant a) A^t E(e^{\tau N} | Z_n \geqslant a) = 1. \quad \ldots\ldots \ (2.2)$$

If t_1 and t_2 are substituted in (2.2), two equations result from which we have

$$E(e^{\tau N}|Z_n \geq \log A) = \frac{A^{t_2}-A^{t_1}}{P(Z_n \geq \log A)[A^{t_2}B^{t_1}-A^{t_1}B^{t_2}]}$$

$$E(e^{\tau N}|Z_n \leq \log B) = \frac{B^{t_1}-B^{t_2}}{P(Z_n \leq \log B)[B^{t_1}A^{t_2}-A^{t_1}B^{t_2}]}$$

Since

$$\phi_N(\tau) = E(e^{\tau N}) = E(e^{\tau N}|Z_n \leq \log B)P(Z_n \leq \log B)$$

$$+E(e^{\tau N}|Z_n \geq \log A)P(Z_n \geq \log A)$$

$$\phi_N(\tau) = \frac{A^{t_2}-A^{t_1}+B^{t_1}-B^{t_2}}{B^{t_1}A^{t_2}-A^{t_1}B^{t_2}}$$

If we consider the limiting case $B \to 0$ and A finite with $E(z) > 0$, then

$$\phi_N(\tau) = \frac{A^{t_2}-A^{t_1}}{B^{t_1}(A^{t_2}-A^{t_1}B^{t_2-t_1})} + \frac{B^{t_1}(1-B^{t_2-t_1})}{B^{t_1}(A^{t_2}-A^{t_1}B^{t_2-t_1})}$$

$$= \frac{A^{t_2}-A^{t_1}}{B^{t_1}(A^{t_2}-A^{t_1}B^{t_2-t_1})} + \frac{1}{A^{t_2}-A^{t_1}B^{t_2-t_1}} - \frac{1}{A^{t_2}/B^{t_2-t_1}-A^{t_1}} .$$

Since $t_2-t_1<0$ and the real part of $t_2<0$ when $E(z)=\mu>0$, it follows that $B^{t_2-t_1}$ and B^{t_2} tend to infinity as $B\to 0$. So the characteristic function of N is

$$\phi_N(\tau) = A^{-t_1} = e^{-at_1} = e^{\frac{a\mu}{\sigma^2}-\frac{a}{\sigma^2}\sqrt{\mu^2-2\sigma^2\tau}}\ ^{1)} \qquad \text{(since } a=\log A\text{)}.$$

The case where B is finite and $A\to 0$, with $\mu<0$ is similarly treated and

$$\phi_N(\tau) = e^{\frac{b\mu}{\sigma^2}-\frac{b}{\sigma^2}\sqrt{\mu^2-2\sigma^2\tau}}$$

The Fourier inversion of the first $\phi_N(\tau)$ gives [2] the inverse normal p.d.f

$$f_N(n) = \frac{ae^{\frac{a\mu}{\sigma^2}}}{\sqrt{2\Pi\sigma^2}}\, e^{-\frac{a^2}{2\sigma^2 n}-\frac{\mu^2 n}{2\sigma^2}}\, n^{-1\frac{1}{2}} \qquad \text{...... (2.3)}$$

2.3 Moments

Let X have inverse normal distribution

1 It is usual to write not the imaginary τ but its real coefficient as arguemtn of ϕ.

2 Wald used differential equations to evaluate the Fourier integral. See his equations in [3a] or [3b, p.192].

$$f_x(x) = \frac{ae^{\frac{a\mu}{\sigma^2}}}{\sqrt{2\Pi\sigma^2}} e^{-\frac{a^2}{2\sigma^2 x}-\frac{\mu^2 x}{2\sigma^2}} x^{-3/2}$$

$$\text{Now } \mu_r' = E(x^r) = \frac{ae^{\frac{a\mu}{\sigma^2}}}{\sqrt{2\Pi\sigma^2}} \int_o^\infty e^{-\frac{a^2}{2\sigma^2 x}-\frac{\mu^2 x}{2\sigma}} x^{-(\frac{3}{2}-r)} dx$$

which is found from tables of Laplace transforms [8] to be

$$\mu_r' = \frac{ae^{\frac{a\mu}{\sigma^2}}}{\sqrt{2\Pi\sigma^2}} 2\left(\frac{a^2}{\mu^2}\right)^{\frac{r-\frac{1}{2}}{2}} K_{r-\frac{1}{2}}\left(\frac{a\mu}{\sigma^2}\right) \qquad \ldots\ldots (2.4)$$

where $K_\nu(2)$ is the modified Bessel function of the second kind [9]. Thus with r=1

$$\mu_1' = \frac{ae^{\frac{a\mu}{\sigma^2}}}{\sqrt{2\Pi\sigma^2}} 2\left(\frac{a^2}{\sigma^2}\right)^{\frac{1}{4}} K_{\frac{1}{2}}\left(\frac{a\mu}{\sigma^2}\right) = \frac{a}{\mu}$$

since $K_{\pm\frac{1}{2}}(z) = e^{-z}\sqrt{\frac{\Pi}{2z}}$. With r=2

$$\mu_2' = E(X^2) = \frac{ae^{\frac{a\mu}{\sigma^2}}}{\sqrt{2\Pi\sigma^2}} 2\left(\frac{a^2}{\sigma^2}\right)^{3/4} K_{\frac{3}{2}}\left(\frac{a\mu}{\sigma^2}\right)$$

$$= \frac{ae^{\frac{a\mu}{\sigma^2}}}{\sqrt{2\Pi\sigma^2}} 2\left(\frac{a^{3/2}}{\mu^{3/2}}\right)\left\{\frac{\sigma^2}{a\mu}K_{\frac{1}{2}}\left(\frac{a\mu}{\sigma^2}\right)+K_{-\frac{1}{2}}\left(\frac{a\mu}{\sigma^2}\right)\right\}$$

$$= \frac{a^2}{\mu^2}(\frac{\sigma^2}{a\mu}+1)$$

since $\frac{2\nu}{z}K_\nu(z) = K_{\nu+1}(z)-K_{\nu-1}(z)$ and $K_{-\nu}(z) = K(z)$ (see [8]).

The variance of X is

$$\sigma_x^2 = \mu_2 = \frac{a\sigma^2}{\mu^3} \qquad \text{...... (2.5)}$$

Moments of order r>2 can be derived in a straight forward manner from (2.4), eg for r=3,4

$$\mu_3' = \frac{3a\sigma^4}{\mu^5}+\frac{3\sigma^2a^2}{\mu^4}+\frac{a^3}{\mu^3}$$

$$\text{...... (2.6)}$$

$$\mu_4' = \frac{15\sigma^6a}{\mu^7}+\frac{15\sigma^4a^2}{\mu^6}+\frac{6\sigma^2a^3}{\mu^5}+\frac{a^4}{\mu^4}$$

thus

$$\mu_3 = \frac{3a\sigma^4}{\mu^5}$$

$$\mu_4 = \frac{15\sigma^6a}{\mu^7}+\frac{3\sigma^4a^2}{\mu^6} \qquad \text{...... (2.7)}$$

$$= 3\mu_2^2+\frac{15\sigma^2a}{\mu^7}$$

A recurrence relation can be found for the origin moments of X by the use of integration by parts as follows

$$\mu'_r = \frac{ae^{\frac{a\mu}{\sigma^2}}}{\sqrt{2\Pi\sigma^2}}\int_0^\infty e^{-\frac{a^2}{2\sigma^2 x}-\frac{\mu^2 x}{2\sigma^2}} x^{-1\frac{1}{2}+r}dx$$

$$= \frac{ae^{\frac{a\mu}{\sigma^2}}}{\sqrt{2\Pi\sigma^2}}\int_0^\infty e^{-\frac{a^2}{2\sigma^2 x}-\frac{\mu^2 x}{2\sigma^2}}\left(\frac{a^2}{2\sigma^2 x^2}-\frac{\mu^2}{2\sigma^2}\right)\frac{2\sigma^2}{a^2}x^{-1\frac{1}{2}+(r+2)}dx$$

$$+\frac{\mu^2}{a^2}\frac{ae^{\frac{a\mu}{\sigma^2}}}{\sqrt{2\pi\sigma^2}}\int_0^\infty e^{-\frac{a^2}{2\sigma^2 x}-\frac{\mu^2 x}{2\sigma^2}} x^{-1\frac{1}{2}+(r+2)}dx$$

$$= \frac{ae^{\frac{a\mu}{\sigma^2}}}{\sqrt{2\Pi\sigma^2}}\left[e^{-\frac{a^2}{2\sigma^2 x}-\frac{\mu^2 x}{2\sigma^2}}\frac{2\sigma^2}{a^2}x^{-1\frac{1}{2}+(r+2)}\right]_0^\infty$$

$$-\frac{ae^{\frac{a\mu}{\sigma^2}}}{\sqrt{2\Pi\sigma^2}}\int_0^\infty e^{-\frac{a^2}{2\sigma^2 x}-\frac{\mu^2 x}{2\sigma^2}}\frac{2\sigma^2}{a^2}\left(r+\frac{1}{2}\right)x^{-1\frac{1}{2}+(r+1)}dx+\frac{\mu^2}{a^2}\mu'_{r+2}$$

$$= 0-\frac{2\sigma^2}{a^2}\left(r+\frac{1}{2}\right)\mu'_{r+1}+\frac{\mu^2}{a^2}\mu'_{r+2}\;.$$

Thus $\mu'_{r+2} = \frac{1}{\mu^2}(a^2\mu'_r+2\sigma^2[r+\frac{1}{2}]\mu'_{r+1})$, and eg.

for $r=0, \mu_2' = \frac{1}{\mu^2}(a^2\mu' + \sigma^2\mu_1') = \frac{a^2}{\mu^2} + \frac{\sigma^2 a}{\mu^3}$ (since $\mu_1' = \frac{a}{\mu}$).

<u>The characteristic function</u>

The characteristic function of X is

$$\phi_x(\theta) = E(e^{\theta x}) = \frac{ae^{\frac{a\mu}{\sigma^2}}}{\sqrt{2\Pi\sigma^2}} \int_0^\infty e^{\theta x - \frac{a^2}{2\sigma^2 x} - \frac{\mu^2 x}{2\sigma^2}} x^{-3/2}\, dx$$

(where $\theta = it$)

$$= \frac{ae^{\frac{a\mu}{\sigma^2}}}{\sqrt{2\Pi\sigma^2}} \int_0^\infty e^{-\frac{a^2}{2\sigma^2 x} - [\frac{\mu^2}{2\sigma^2} - \theta]x} x^{-3/2}\, dx$$

$$= \frac{ae^{\frac{a\mu}{\sigma^2}}}{\sqrt{2\Pi\sigma^2}} \Bigg/ \frac{ae^{\frac{a}{\sigma^2}\sqrt{\mu^2 - 2\sigma^2\theta}}}{\sqrt{2\Pi\sigma^2}}$$

$$= e^{\frac{a\mu}{\sigma^2} - \frac{a}{\sigma^2}\sqrt{\mu^2 - 2\sigma^2\theta}}$$

Thus if $Y = \sum_{i=1}^{n} X_i$ is the sum of n independent variables having inverse normal distribution,

$$\phi_y(\theta) = [\phi_x(\theta)]^n = e^{\frac{na\mu}{\sigma^2} - \frac{na}{\sigma^2}\sqrt{\mu^2 - 2\sigma^2\theta}}$$

$$\text{hence } f_y(y) = \frac{nae^{\frac{na\mu}{\sigma^2}}}{\sqrt{2\Pi\sigma^2}} e^{-\frac{n^2a^2}{2\sigma^2 y} - \frac{\mu^2 y}{2\sigma^2}} y^{-1\frac{1}{2}}$$

$$\text{and } f_{\bar{x}}(u) = \frac{\sqrt{n}ae^{\frac{na\mu}{\sigma^2}}}{\sqrt{2\Pi\sigma^2}} e^{-\frac{na^2}{2\sigma^2 u} - \frac{n\mu^2 u}{2\sigma^2}} u^{-1\frac{1}{2}} \qquad \ldots\ldots (2.8)$$

Fisher's coefficient of skewness is, from (2.7) and (3.5),

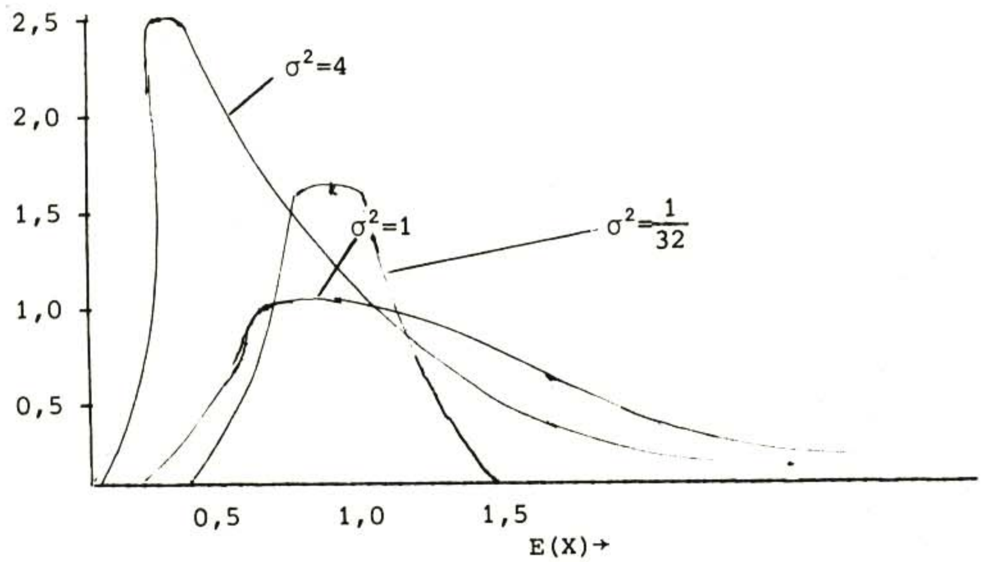
$$\gamma_1 = \frac{\mu_3}{\mu_2^{3/2}} = \frac{3\sigma}{\sqrt{a\mu}}$$

which tends to zero as $\sigma^2 \to 0$, i.e. the p.d.f of X seems to tend to normality as $\sigma^2 \to 0$. Tweedie has drawn the curves which appear in figure 1 for different values of σ^2 with $\mu=1$ for the p.d.f

$$f_x(x) = \frac{1}{\sqrt{2\Pi\sigma^2}} e^{-\frac{1}{2}\frac{(x-\mu)^2}{x\sigma^2}} x^{-3/2}$$

which is of the same form as the one considered in this essay except for some constants.

Figure 1: Probability density curves of X



It is seen from figure 1 that as σ^2 becomes smaller and smaller, the curves tend to be symmetric about $\mu=1$.

Fisher's skewness, $\gamma_1 = \frac{\mu_3}{\mu_2^{3/2}}$ above, may have any value between $-\infty$ and ∞, which is not very informative. We now use Jensen's inequality [10] to redefine skewness. The inequality states that $E(g[x]) \geq g[E(x)]$ if g is convex. If we work with $(X-\mu)^3$ instead of X and let $g = (\quad)^{4/3}$, which is convex, then by the above inequality $E([(X-\mu)^3]^{\frac{4}{3}}) \geq E^{4/3}[(X-\mu)]$ i.e. $\mu_4 \geq \mu_3^{4/3}$. Thus if we redefine $Sk = \frac{\mu_3}{\mu^{3/4}}$, it follows that $-1 \leq Sk \leq 1$ (with which we compare $-1 \leq r \leq 1$, where r is the correlation coefficient). Without loss of generality we shall assume Tweedie's form of the inverse normal distribution i.e. with a=1 in our case. Then

$$(Sk)^4 = \frac{\sigma^4 (\frac{3}{\mu^7})^4}{(\frac{15}{\mu^2}\sigma^2 + \frac{3}{\mu^6})^3}$$

Consider the curves as given in figure 1, with $\mu=1$ for each case, we have that if

(i) $\sigma^2=1$, $Sk = 0,0130$

(ii) $\sigma^2=\frac{1}{8}$, $Sk = 0,0101$

(iii) $\sigma^2=\frac{1}{32}$, $Sk = 0,00189$

The distribution is unimodal since $X>0$ and has its mode at

$$X = \frac{a}{\mu}\{\sqrt{\frac{9\sigma^4}{4a^2}+1} - \frac{3\sigma^2}{2\mu^2}\}$$

which tends to $\frac{a}{\mu}$ as $\sigma^2 \to 0$. Hence Karl Pearson's coefficient of skewness $Sk = \frac{\text{mean-mode}}{\sigma_x} \to 0$ as $\sigma^2 \to 0$.

The median and other quantities cannot be expressed in closed form because of the integration by parts involved in their calculation (which leads to infinite series).

Chapter 3

ESTIMATION AND TESTS OF HYPOTHESES REGARDING THE PARAMETERS OF THE INVERSE NORMAL DISTRIBUTION

3.1 Method of moments

Since the mean of X is $\frac{a}{\mu}$, and therefore the mean of the first sample moment $M_1 = \bar{X}$ is $\frac{a}{\mu}$, the moments estimator for μ is $\hat{\mu} = \frac{a}{\bar{X}} = \frac{a}{M_1}$. The population variance is $\frac{a\sigma^2}{\mu^3}$ and the sample variance is $M_2' - (M_1')^2$ and

$$E(M_2' - (M_1')^2) = \frac{\sigma^2 a}{\mu^3}\left(\frac{n-1}{n}\right)$$

so the moments estimator of σ^2 is

$$\widehat{\sigma^2} = \frac{n}{(n-1)} \frac{a^2}{(M_1')^3} \left(M_2' - [M_1']^2\right)$$

3.2 Maximum likelihood estimates

The likelihood function of the random sample $X_1, \ldots, X_n$ is

$$L(\mu,\sigma^2) = \frac{a^n e^{\frac{na\mu}{\sigma^2}}}{(2\Pi\sigma^2)^{\frac{n}{2}}} e^{-\frac{a^2}{2\sigma^2}\Sigma\frac{1}{x}-\frac{\mu^2}{2\sigma^2}\Sigma x} \Pi x^{-3/2}$$

and $K(\mu,\sigma^2) \equiv \ln L(\mu,\sigma^2) = n \ln a - \frac{n}{2}\ln(2\Pi\sigma^2) + \frac{na\mu}{\sigma^2}$

$$- \frac{a^2}{2\sigma^2}\Sigma\frac{1}{x} - \frac{\mu^2}{2\sigma^2}\Sigma x - \frac{3}{2}\Sigma \ln x. \qquad \text{...... (3.1)}$$

Differentiating (3.1) with respect to a, μ and σ^2 and equating to zero, leads to *only* two linearly independent equations viz.

$$\frac{\partial K}{\partial \mu} = \frac{na}{\sigma^2} - \frac{\mu}{\sigma^2}\Sigma x = 0 \qquad \text{...... (3.2)}$$

$$\frac{\partial K}{\partial \sigma^2} = -\frac{n}{2\sigma^2} - \frac{na\mu}{\sigma^4} + \frac{a^2}{2\sigma^4}\Sigma\frac{1}{x} + \frac{\mu^2}{2\sigma^4}\Sigma x = 0 \qquad \text{...... (3.3)}$$

$$\left(\text{and } \frac{\partial K}{\partial a} = \frac{n}{a} + \frac{n\mu}{\sigma^2} - \frac{a}{\sigma^2}\ \frac{1}{x}\Sigma = 0\right), \qquad \text{...... (3.4)}$$

since from (3.2) $\mu = \frac{na}{\Sigma x}$, and substituting into (3.3) and (3.4) we have

$$-\frac{n}{2\sigma^2} - \frac{(na)^2}{\sigma^4 \Sigma x} + \frac{a^2}{2\sigma^4}\Sigma\frac{1}{x} + \frac{(na)^2}{2\sigma^4 \Sigma x^2}\Sigma x = 0$$

i.e. $-\frac{n}{2} - \frac{(na)^2}{2\sigma^2 \Sigma x} + \frac{a^2}{2\sigma^2}\Sigma\frac{1}{x} = 0$

i.e. $\frac{n}{a} + \frac{n^2 a}{\Sigma x} - \frac{a}{2\sigma^2}\Sigma\frac{1}{x} = 0,$ ...... (3.5)

and from (3.4)

$$\frac{n}{a} + \frac{n^2 a}{\Sigma x} - \frac{a}{\sigma^2}\Sigma\frac{1}{x} = 0, \qquad \text{...... (3.6)}$$

which is the same as (3.5) above, the reason for this being that in the case under consideration and the case of Wald, a is *given* and cannot be determined by the maximum likelihood method. (3.2) and (3.3) lead to

$$\hat{\mu} = \frac{na}{\Sigma x} = \frac{a}{\bar{x}} \quad \text{and}$$

$$\widehat{\sigma^2} = a^2\left(\frac{1}{n}\Sigma\frac{1}{x} - \frac{n}{\Sigma x}\right) = a^2\left(\frac{1}{\tilde{x}} - \frac{1}{\bar{x}}\right)$$

where $\tilde{x} = \frac{1}{\frac{1}{n}\Sigma\frac{1}{x}}$, is the harmonic mean of the x_i. From (2.8) we have that

$$f_{\hat{\mu}}(t) = \sqrt{\frac{na}{2\Pi\sigma^2}}\, e^{\frac{na\mu}{\sigma^2}}\, e^{-\frac{nat}{2\sigma^2}-\frac{n\mu^2 a}{2\sigma^2 t}}\, t^{-\frac{1}{2}}$$

where $t = \frac{na}{\Sigma x}$.

For the sample

$$f_{x_1,\ldots,x_n}(x_1,\ldots,x_n) = \frac{a^n e^{\frac{na\mu}{\sigma^2}}}{(2\Pi\sigma^2)^{\frac{n}{2}}} e^{-\frac{a^2}{2\sigma^2}\Sigma\frac{1}{x}-\frac{\mu^2}{2\sigma^2}\Sigma x} \Pi x^{-3/2}$$

$$= \sqrt{\frac{na}{2\Pi\sigma^2}}\, e^{\frac{na\mu}{\sigma^2}} e^{-\frac{nat}{2\sigma^2}-\frac{n\mu^2 a}{2\sigma^2 t}} t^{-\frac{1}{2}}$$

$$\frac{a^{n-\frac{1}{2}}}{n^{\frac{1}{2}}(2\Pi\sigma^2)^{\frac{n-1}{2}}} e^{\frac{(na)^2}{2\sigma^2\Sigma x}-\frac{a^2}{2\sigma^2}\Sigma\frac{1}{x}} (\Sigma x)^{-\frac{1}{2}} \Pi x^{-3/2} ,$$

from which because of the Fisher-Neyman criterion, it follows that $\hat{\mu}$ is a sufficient statistic for μ.

The distribution of $\widehat{\sigma^2}$

We are now going to give a derivation by Tweedie of the distribution of $\widehat{\sigma^2}$. The results are of course well-known from normal sampling theory but the derivation is interesting and possibly the inverse normal distribution may derive from some other distribution than the normal and may even be a non-sampling distribution.

Consider the Laplace transform of the p.d.f. of $\widehat{\sigma^2}$, i.e.

$$E(e^{\alpha\widehat{\sigma^2}}) = \int_{-\infty}^{\infty}\cdots\int e^{-\alpha\widehat{\sigma^2}} f_{\hat{\mu}}(t) \frac{a^{n-\frac{1}{2}}}{n^{\frac{1}{2}}(2\Pi\sigma^2)^{\frac{n-1}{2}}} e^{\frac{(na)^2}{2\sigma^2\Sigma x}-\frac{a^2}{2\sigma^2}\Sigma\frac{1}{x}} (\Sigma x)^{-\frac{1}{2}}\Pi(x^{-3/2}\, dx)$$

$$= \int_{\hat{\mu}=o}^{\hat{\mu}=\infty} f_{\hat{\mu}}(t) \int_{o}^{\infty} e^{-\alpha\widehat{\sigma^2}} \frac{a^{n-\frac{1}{2}}}{n^{\frac{1}{2}}(2\pi\sigma^2)^{\frac{n-1}{2}}} e^{\frac{(na)^2}{2\sigma^2\Sigma x}-\frac{a^2}{2\sigma^2}\Sigma\frac{1}{x}} (\Sigma x)^{-\frac{1}{2}}\Pi(x^{-3/2}\, dx)$$

where the multiple integral is seen not to contain μ. From the form of $f_{\hat{\mu}}(t)$, and the fact that it has a unique Laplace transform, it follows that derivative of the multiple integral with respect to t is the Laplace transform of the conditional p.d.f. of $\widehat{\sigma^2}$ given t. Therefore

$$\frac{\partial}{\partial t}\int_{o}^{\infty}\cdots\int e^{-\alpha\widehat{\sigma^2}} \frac{a^{n-\frac{1}{2}}}{n^{\frac{1}{2}}(2\Pi\sigma^2)^{\frac{n-1}{2}}} e^{\frac{(na)^2}{2\sigma^2\Sigma x}-\frac{a^2}{2\sigma^2}\Sigma\frac{1}{x}} (\Sigma x)^{-\frac{1}{2}}\Pi(x^{-3/2}\, dx)$$

$$= E(e^{-\alpha\widehat{\sigma^2}}\,|\,t)$$

$$\text{i.e. } E(e^{-\alpha\widehat{\sigma^2}}\,|\,t) = \frac{\partial}{\partial t}\int_{o}^{\infty}\cdots\int \frac{a^{n-\frac{1}{2}}}{n^{\frac{1}{2}}(2\Pi\sigma^2)^{\frac{n-1}{2}}} e^{-\widehat{\sigma^2}(\alpha+\frac{a^2 n}{2\sigma^2})} (\Sigma x)^{-\frac{1}{2}}\Pi(x^{-3/2}\, dx)$$

$$\left(\text{since } a^2\left[\frac{1}{n}\Sigma\frac{1}{x} - \frac{n}{\Sigma x}\right] = \widehat{\sigma^2}\right) \qquad \ldots\ldots (3.7)$$

To evaluate (3.7), first put $\alpha=0$, then

$$\frac{\partial}{\partial t}\int_0^\infty \!\!\cdots\!\int e^{-\frac{\widehat{\sigma^2} a^2 n}{2\sigma^2}} (\Sigma x)^{-\frac{1}{2}} \Pi (x^{-3/2}\, dx) = \frac{n^{\frac{1}{2}} (2\Pi\sigma^2)^{\frac{n-1}{2}}}{a^{n-\frac{1}{2}}} \quad . \qquad \ldots\ldots \ (3.8)$$

On the right hand side of (3.8) put $\frac{1}{\sigma^2} = (\frac{2\alpha}{a^2 n} + \frac{1}{\sigma^2})$, then

$$\frac{\partial}{\partial t}\int_0^\infty \!\!\cdots\!\int \frac{a^{n-\frac{1}{2}}}{n^{\frac{1}{2}}(2\Pi\sigma^2)^{\frac{n-1}{2}}} e^{-\widehat{\sigma^2}(\alpha+\frac{a^2 n}{2\sigma^2})} (\Sigma x)^{-\frac{1}{2}} \Pi (x^{-3/2}\, dx)$$

$$= \frac{a^{n-\frac{1}{2}}}{n^{\frac{1}{2}}(2\Pi\sigma^2)^{\frac{n-1}{2}}} \qquad \frac{n^{\frac{1}{2}}\left(2\Pi\frac{1}{\frac{2\alpha}{a^2 n}+\frac{1}{\sigma^2}}\right)^{\frac{n-1}{2}}}{a^{n-\frac{1}{2}}}$$

$$= \frac{1}{\left(\frac{2\alpha\sigma^2}{a^2 n}+1\right)^{\frac{n-1}{2}}}$$

If we put $\alpha=-i\theta$, then

$E(e^{i\theta\widehat{\sigma^2}}\,|\,t) = (1-\frac{2i\theta\sigma^2}{a^2 n})^{-\frac{n-1}{2}}$, which is the characteristic function of a χ^2_{n-1} variate, i.e.

$$\frac{\sigma^2}{a^2 n}\chi^2_{n-1} = \widehat{\sigma^2} \ .$$

The p.d.f. of $Y = \frac{\widehat{\sigma^2}}{a^2}$ is

$$f_y\left(\frac{\widehat{\sigma^2}}{a^2}\right) = \frac{1}{2^{\frac{n-1}{2}}\Gamma\left(\frac{n-1}{2}\right)} e^{-\frac{n}{2\sigma^2}\widehat{\sigma^2}} \left(\frac{n\widehat{\sigma^2}}{\sigma^2}\right)^{\frac{n-3}{2}} \frac{a^4 n}{\sigma^2}$$

The joint p.d.f. of $X_1, .., X_n$ may be written as

$$\underset{X_1,\ldots,X_n}{f(x_1,..,x_n)} = \frac{a^n e^{\frac{na\mu}{\sigma^2}}}{(2\Pi\sigma^2)^{\frac{n}{2}}} e^{-\frac{a^2}{2\sigma^2}\Sigma\frac{1}{x} - \frac{\mu^2}{2\sigma^2}\Sigma x} \Pi x^{-3/2}$$

$$= \sqrt{\frac{na}{2\Pi\sigma^2}} e^{\frac{na\mu}{\sigma^2}} e^{-\frac{nat}{\sigma^2} - \frac{n\mu^2 a}{2\sigma^2 t}} t^{-\frac{1}{2}}$$

$$\cdot \frac{1}{2^{\frac{n-1}{2}}\Gamma\left(\frac{n-1}{2}\right)} e^{-\frac{n}{2\sigma^2}\sigma^2} \left(\frac{n\widehat{\sigma^2}}{\sigma^2}\right)^{\frac{n-3}{2}} \frac{a^4 n}{\sigma^2}$$

$$\cdot \frac{a\Gamma\left(\frac{n-1}{2}\right)}{(\Pi)^{\frac{n-1}{2}}} \left(\Sigma\frac{1}{x} - \frac{n^2}{\Sigma x}\right)^{-\frac{n-3}{2}} \Pi x^{-3/2}$$

i.e. $f(x_1,..,x_n;\mu,\sigma^2) = f_{\hat{\mu}}^{(t)} f_y\left(\frac{\widehat{\sigma^2}}{a^2}\right) h(x_1,..,x_n)$

Where $h(x_1,..,x_n)$ is a function which does not depend upon μ and σ^2 jointly or separately. Moreover, the conditional p.d.f. of $\frac{\widehat{\sigma^2}}{a^2}$ is independent of μ, hence $\hat{\mu}$ and $\widehat{\sigma^2}$ are joint sufficient statistics for μ and σ^2 respectively.

3.3 Tests of hypotheses

3.3.1 Best tests

(i) for μ: If it is desired to test the simple hypothesis $H_o:\mu=\mu_o$ against $H_1:\mu=\mu_1$ with σ^2 known, then from the Neyman-Pearson theorem

$$\frac{L(\mu_o)}{L(\mu_1)} = e^{\frac{na}{\sigma^2}(\mu_o-\mu_1)-\frac{n\bar{x}}{2\sigma^2}(\mu_o-\mu_1)} \leqslant k$$

i.e. if $\bar{x} \geqslant c$ where c is such that $P(\bar{X} \geqslant c)=\alpha$ is the desired size of the test. Since by (2.8)

$$f_{\bar{x}}(\bar{x}) = \frac{a\sqrt{n}e^{\frac{na}{\sigma^2}}}{\sqrt{2\Pi\sigma^2}} e^{-\frac{na^2}{2\sigma^2\bar{x}}-\frac{n\mu^2\bar{x}}{2\sigma^2}} \bar{x}^{-1\frac{1}{2}}$$

we have that

$$P(\bar{X} \geqslant c) = \frac{\sqrt{n}ae^{\frac{na\mu}{\sigma^2}}}{\sqrt{2\Pi\sigma^2}} \int_o^\infty e^{-\frac{na^2}{2\sigma^2\bar{x}}-\frac{n\mu^2\bar{x}}{2\sigma^2}} \bar{x}^{-1\frac{1}{2}} d\bar{x}$$

which could be evaluated if the distribution of $\mathbf{X}$ were tabulated.

(ii) <u>for σ^2:</u>

Let $H_o:\sigma^2=\sigma_o^2$, $H_1:\sigma^2=\sigma_1^2$

then $$\frac{L(\sigma_o^2)}{L(\sigma_1^2)} = e^{(na\mu-\frac{a}{2}\Sigma\frac{1}{x}-\frac{\mu^2}{2}\Sigma x)(\frac{1}{\sigma^2}-\frac{1}{\sigma_1^2}} (\frac{\sigma_1}{\sigma_o})^n \leqslant k$$

i.e. $$(na\mu-\frac{a^2}{2}\Sigma\frac{1}{x}-\frac{\mu^2}{2}\Sigma x)(\frac{1}{\sigma_o^2}-\frac{1}{\sigma_1^2}) \leqslant \ln k - n\ln(\frac{\sigma_1}{\sigma_o})$$

i.e. $$\frac{a^2}{2\bar{x}}+\frac{\mu^2\bar{x}}{2} > c$$

The characteristic function of

$$\frac{a^2}{2x\sigma^2}+\frac{\mu^2 x}{2\sigma^2} = V \quad \text{is}$$

$$\phi_V(\alpha) = \frac{ae^{\frac{a\mu}{\sigma^2}}}{\sqrt{2\Pi\sigma^2}}\int_o^\infty e^{[\frac{a^2}{2x}+\frac{\mu^2 x}{2}]\frac{i\alpha}{\sigma^2}-\frac{a^2}{2\sigma^2 x}-\frac{\mu^2 x}{2\sigma^2}} x^{-3/2}\, dx$$

$$= \frac{ae^{\frac{a\mu}{\sigma^2}}}{\sqrt{2\Pi\sigma^2}} \int_0^\infty e^{-\frac{\mu^2 x}{2\sigma^2}(1-i\alpha)-\frac{a^2}{2\sigma^2 x}(1-i\alpha)} x^{-3/2}\, dx$$

$$= e^{\frac{a\mu i\alpha}{\sigma^2}} (1-i\alpha)^{-\frac{1}{2}} ,$$

$$\text{so } \phi_{\Sigma V}^{(\alpha)} = e^{\frac{na\mu i\alpha}{\sigma^2}} (1-i\alpha)^{-\frac{n}{2}} ,$$

and the Fourier inverse of $\phi_{\Sigma V=U}^{(\alpha)}$,

$$f_U(u) = \frac{1}{2\Pi} \int_{-\infty}^{\infty} e^{-i\alpha u}\, e^{\frac{na\mu i\alpha}{\sigma^2}} (1-i\alpha)^{-\frac{n}{2}} d\alpha$$

$$= \frac{1}{2\Pi} \int_{-\infty}^{\infty} e^{-i\alpha(u-\frac{na\mu}{\sigma^2})} (1-i\alpha)^{-\frac{n}{2}} da$$

$$= \frac{1}{\Gamma(\frac{n}{2})} e^{-(u-\frac{na\mu}{\sigma^2})} (u-\frac{na\mu}{\sigma^2})^{\frac{n}{2}-1} .$$

Let $W = \frac{\sigma^2 U}{n}$

$$\text{so } f_W(w) = \frac{\left(\frac{n}{\sigma^2}\right)^{\frac{n}{2}}}{\Gamma\left(\frac{n}{2}\right)} e^{-\frac{n(w-a\mu)}{\sigma^2}} (w-a\mu)^{\frac{n}{2}-1}$$

Thus we reject H_o if

$$P(W \geq c) = \frac{\left(\frac{n}{\sigma^2}\right)^{\frac{n}{2}}}{\Gamma\left(\frac{n}{2}\right)} \int_c^{\infty} e^{-\frac{n(w-a\mu)}{\sigma^2}} (w-a\mu)^{\frac{n}{2}-1} dw$$

$$\text{i.e. if } \frac{1}{\Gamma\left(\frac{n}{2}\right)} \int_{\frac{n(c-a\mu)}{\sigma^2}}^{\infty} e^{-t} t^{\frac{n}{2}-1} dt = \alpha$$

where $t = \frac{n(w-a\mu)}{\sigma^2}$.

3.3.2 The generalized likelihood ratio test

Consider two composite hypotheses

$H_o: \mu=\mu_o$, $\sigma^2>0$ unspecified and
$H_1: \mu\neq\mu_o$, σ^2 unspecified. The likelihood ratio is

$$\lambda = \frac{\max_{\sigma^2} L(\mu_o,\sigma^2)}{\max_{\mu,\sigma^2} L(\mu,\sigma^2)}$$

$$= \left[\frac{a^2(\frac{1}{\tilde{x}}-\frac{1}{\bar{\bar{x}}})}{2\mu_o a-\frac{a^2}{\tilde{x}}-\mu_o^2\bar{\bar{x}}}\right]^{\frac{n}{2}} .$$

In the case of the hypotheses $H_o:\sigma^2=\sigma_o^2$ μ unspecified and $H_1:\sigma^2\neq\sigma_o^2$, μ unspecified,

$$\lambda = \frac{\max_{\mu} L(\mu,\sigma_o^2)}{\max_{\mu,\sigma^2} L(\mu,\sigma^2)}$$

$$=\left(\frac{a^2[\frac{1}{\tilde{x}}-\frac{1}{\bar{\bar{x}}}]}{\sigma_o^2}e^{-a^2(\frac{1}{\tilde{x}}-\frac{1}{\bar{\bar{x}}})+1}\right)^{\frac{n}{2}} .$$

We see that in both instances the likelihood ratio test leads to more difficult sampling theory than the best test.

Chapter 4

SAMPLING FROM THE NORMAL DISTRIBUTION AS A RENEWAL PROCESS AND DISTRIBUTIONS RELATED TO THE INVERSE NORMAL DISTRIBUTION

4.1 Sampling from the normal distribution as a renewal process

Consider the p.d.f.

$$f_N(n) = \frac{ae^{\frac{a\mu}{\sigma^2}}}{\sqrt{2\Pi\sigma^2}} e^{-\frac{a^2}{2\sigma^2 n}-\frac{\mu^2 n}{2\sigma^2}} n^{-1\frac{1}{2}}, n\geqslant 0$$

where $a=\Sigma x$ and the X's are items of a random sample from a normal distribution $n(\mu,\sigma^2)$. The random variable N may be regarded as the number of renewals in a renewal process for the following reason: the values X may be regarded as the 'inter-event times', ΣX the 'waiting time' up to the n-th event (or 'first passagetime'), a the 'time t' and the sample size n 'the number of events'. The moments of N for the above distribution therefore follow those of N_t, the number of renewals in time t of a renewal process. For the above distribution, we have that

$$\mu_1' = \frac{a}{\mu}\ , \ \mu_2 = \frac{\sigma^2 a}{\mu^3} \qquad \text{...... (4.1)}$$

whereas in a renewal process we have (see Cox[11])

$$E(N_t) = \frac{t}{\mu} + \frac{\sigma^2-\mu^2}{2\mu^2} + 0(1), \qquad \ldots\ldots (4.2)$$

$$var(N_t) = \frac{\sigma^2 t}{\mu^2} + \frac{1}{12} + \frac{5\sigma^4}{4\mu^4} - \frac{2\mu_3}{3\mu^3} + 0(1)$$

where $\mu=E(X)$, $\sigma^2=var(X)$, $\mu_3=E(X-\mu)^3$ etc. The simplification of (4.2) to (4.1) must be due to the fact that for the normal distribution

$$\mu_{2n+1} = 0 \text{ and } \mu_{2n} = \frac{(2n-1)!\sigma^{2n}}{2^{n-1}(n-1)!}, \quad n=1,2,3,\ldots,$$

i.e. μ_{2n} is a function of σ only.

4.2 Distributions related to the inverse normal distribution

Consider the p.d.f. of a random variable X

$$f_X(x) = \frac{\mu e^{\frac{a\mu}{\sigma^2}}}{\sqrt{2\Pi\sigma^2}} e^{-\frac{a^2}{2\sigma^2 x}-\frac{\mu^2 x}{2\sigma^2}} x^{-\frac{1}{2}}, x\geq 0.$$

Its characteristic function is

$$\phi_x(\alpha) = \frac{\mu e^{\frac{\mu a}{\sigma^2}}}{\sqrt{2\Pi\sigma^2}} \int_0^\infty e^{i\alpha x - \frac{a^2}{2\sigma^2 x} - \frac{\mu^2 x}{2\sigma^2}} x^{-\frac{1}{2}} dx$$

$$= \frac{\mu e^{\frac{\mu a}{\sigma^2}}}{\sqrt{2\Pi\sigma^2}} \int_0^\infty e^{-\frac{a^2}{2\sigma^2 x} - \frac{x}{2\sigma^2}(\mu^2 - 2i\alpha\sigma^2)} x^{-\frac{1}{2}} dx$$

$$= \mu e^{\frac{\mu a}{\sigma^2} - \frac{a^2}{\sigma^2}\sqrt{\mu^2 - 2i\alpha\sigma^2}} \qquad \frac{1}{\sqrt{\mu^2 - 2i\alpha\sigma^2}}$$

The fact that in the exponential of the p.d.f., x and $\frac{1}{x}$ appear, makes it natural to inquire after the joint distribution of the arithmetic and harmonic means of a random sample.

Let $\Sigma X = Y$ and $\Sigma\frac{1}{x} = Z$. The joint characteristic function of Y and Z is

$$\phi_{y,z}(\alpha,\beta) = \frac{\mu^n e^{\frac{na\mu}{\sigma^2}}}{(2\Pi\sigma^2)^{\frac{n}{2}}} \int_0^\infty e^{i\alpha\Sigma x + i\beta\Sigma\frac{1}{x} - \frac{a^2}{2\sigma^2}\Sigma\frac{1}{x} - \frac{\mu^2}{2\sigma^2}\Sigma x} \Pi(x^{-\frac{1}{2}} dx)$$

$$= \left[\frac{\mu e^{\frac{a\mu}{\sigma^2}}}{\sqrt{2\Pi\sigma^2}} \int_0^\infty e^{i\alpha x + \frac{i\beta}{x} - \frac{a^2}{2\sigma^2 x} - \frac{\mu^2 x}{2\sigma^2}} x^{-\frac{1}{2}} dx \right]^n$$

$$= \mu^n e^{\frac{na\mu}{\sigma^2} - \frac{n}{\sigma^2}\sqrt{\mu^2 - 2i\alpha\sigma^2}\sqrt{a^2 - 2i\beta\sigma^2}} (\mu^2 - 2i\alpha\sigma^2)^{-\frac{n}{2}}$$

If we invert following Wald (see (2.3) above), we have

$$f_{Y,Z}(y,z) = \frac{\mu^n e^{\frac{na\mu}{\sigma^2}}}{(2\Pi)^2} \iint_{-\infty}^{\infty} e^{-\frac{n}{\sigma^2}\sqrt{\mu^2 - 2i\alpha\sigma^2}\sqrt{a^2 - 2i\beta\sigma^2} - i\beta - i\alpha y}$$

$$(\mu^2 - 2i\alpha\sigma^2)^{-\frac{n}{2}} d\beta d\alpha$$

$$= \frac{\mu^n e^{\frac{na\mu}{\sigma^2}}}{2\Pi} \int_{-\infty}^{\infty} \frac{n\sqrt{\mu^2 - 2i\alpha\sigma^2}}{\sqrt{2\Pi\sigma^2}} e^{-\frac{n^2(\mu^2 - 2i\alpha\sigma^2)}{2\ z} - \frac{a^2 z}{2\sigma^2}} z^{-1\frac{1}{2}}$$

$$\cdot e^{-i\alpha y} (\mu^2 - 2i\alpha\sigma^2)^{-\frac{n}{2}} d\alpha$$

$$= \frac{n\mu^n e^{\frac{na\mu}{\sigma^2}}}{\sqrt{2\Pi\sigma^2}} e^{-\frac{n^2\mu^2}{2\sigma^2 z} - \frac{a^2 z}{2\sigma^2}} z^{-1\frac{1}{2}}$$

$$\frac{\mu^{-(n-1)}}{2\Pi}\int_{-\infty}^{\infty} e^{-i\alpha(y-\frac{n^2}{z})}(1-\frac{2i\alpha\sigma^2}{\mu^2})^{\frac{n-1}{2}}dx$$

$$= \frac{n\mu e^{\frac{na\mu}{\sigma^2}}}{\sqrt{2\Pi\sigma^2}} e^{-\frac{n^2\mu^2}{2\sigma^2 z}-\frac{a^2 z}{2\sigma^2}} z^{-1\frac{1}{2}}$$

$$\cdot \frac{(\frac{\mu^2}{2\sigma^2})^{\frac{n-1}{2}}}{\Gamma(\frac{n-1}{2})} e^{-\frac{\mu^2}{2\sigma^2}(y-\frac{n^2}{z})} (y-\frac{n^2}{Z})^{\frac{n-3}{2}} \quad .$$

If we put $U=\frac{Y}{n}=\bar{X}$ (the sample AM) and $V=\frac{n}{z}=\tilde{X}$ (the sample HM), then $y=nu$ and $z=\frac{n}{v}$, and the Jacobian of the transformation is $\frac{n^2}{v^2}$, so

$$f_{U,V}(u,v) = \frac{n\mu e^{\frac{na\mu}{\sigma^2}}}{\sqrt{2\Pi\sigma^2}} e^{-\frac{n\mu^2 v}{2\sigma^2}-\frac{na^2}{2\sigma^2 v}} (\frac{n}{v})^{-1\frac{1}{2}}$$

$$\cdot \frac{(\frac{\mu^2}{2\sigma^2})^{\frac{n-1}{2}}}{\Gamma(\frac{n-1}{2})} e^{-\frac{n\mu^2}{2\sigma^2}(u-v)} n^{\frac{n-3}{2}} (u-v)^{\frac{n-3}{2}} \frac{n^2}{v^2}$$

$$= \frac{\sqrt{n}\mu e^{\frac{na\mu}{\sigma^2}}}{\sqrt{2\Pi\sigma^2}} e^{-\frac{n\mu^2 v}{2\sigma^2}-\frac{na^2}{2\sigma^2 v}} v^{-\frac{1}{2}}$$

$$\cdot \frac{(\frac{n\mu^2}{2\sigma^2})^{\frac{n-1}{2}}}{\Gamma(\frac{n-1}{2})} e^{-\frac{n\mu^2}{2\sigma^2}(u-v)} (u-v)^{\frac{n-3}{2}}$$

Let $U-V=\bar{X}-\tilde{X}=W$, i.e. $U=V+W$ and $\frac{du}{dw}=1$, so

$$f_{W,V}(w,v) = \frac{\sqrt{n}\mu e^{\frac{na\mu}{\sigma^2}}}{\sqrt{2\Pi\sigma^2}} e^{-\frac{n\mu^2 v}{2\sigma^2}-\frac{na^2}{2\sigma^2 v}} v^{-\frac{1}{2}}$$

$$\cdot \frac{\left(\frac{n}{2}\right)^{\frac{n-1}{2}}}{\Gamma\left(\frac{n-1}{2}\right)} e^{-\frac{na}{2}w} w^{\frac{n-3}{2}} \qquad \ldots\ldots (4.3)$$

In this case $W=\bar{X}-\tilde{X}$ and $V=\tilde{X}$ are independent, W being a gamma variate.

For the inverse normal distribution

$$f_X(x) = \frac{ae^{\frac{a\mu}{\sigma^2}}}{\sqrt{2\Pi\sigma^2}} e^{-\frac{a^2}{2\sigma^2 x}-\frac{\mu^2 x}{2\sigma^2}} x^{-1\frac{1}{2}} \quad , \; x\geqslant 0$$

it follows as above that

$$\phi_{\Sigma X, \Sigma\frac{1}{X}}(\alpha,\beta) = \frac{a^n}{(a^2-2i\beta\sigma^2)^{\frac{n}{2}}} e^{\frac{na\mu}{\sigma^2}-\frac{n}{\sigma^2}\sqrt{a^2-2i\beta\sigma^2}\sqrt{\mu^2-2i\alpha\sigma^2}}$$

On inverting, again following Wald, we have

$$f_{\Sigma x, \Sigma \frac{1}{x}}(y,z) = \frac{a^n e^{\frac{na\mu}{\sigma^2}}}{(2\Pi)^2} \int\int_{-\infty}^{\infty} e^{-\frac{n}{\sigma^2}\sqrt{\mu^2-2i\alpha\sigma^2}\sqrt{a^2-2i\beta\sigma^2}\ -i\alpha y-i\beta z}$$

$$(a^2-2i\beta\sigma^2)\,d\alpha d\beta$$

$$= \frac{a^n e^{\frac{na\mu}{\sigma^2}}}{2\Pi} \int_{-\infty}^{\infty} \frac{n\sqrt{a^2-2i\beta\sigma^2}}{\sqrt{2\Pi\sigma^2}} e^{-\frac{n^2(a^2-2\sigma^2 i\beta)}{2\sigma^2 y}-\frac{\mu^2 y}{2\sigma^2}}$$

$$\cdot\ y^{-1\frac{1}{2}} e^{-i\beta z}(a^2-2i\beta\sigma^2)^{-\frac{n}{2}} d\beta$$

$$= \frac{na^n e^{\frac{na\mu}{\sigma^2}}}{\sqrt{2\Pi\sigma^2}} e^{-\frac{n^2a^2}{2\sigma^2 y}-\frac{\mu^2 y}{2\sigma^2}} y^{-1\frac{1}{2}}$$

$$\cdot\ \frac{1}{2\Pi}\int_{-\infty}^{\infty} e^{-i\beta(z-\frac{n^2}{y})}(a^2-2i\beta\sigma^2)^{-\frac{n-1}{2}} d\beta$$

$$= \frac{na^n e^{\frac{na\mu}{\sigma^2}}}{\sqrt{2\Pi\sigma^2}} e^{-\frac{n^2a^2}{2\sigma^2 y}-\frac{\mu^2 y}{2\sigma^2}} y^{-1\frac{1}{2}}$$

$$\frac{a^{-(n-1)}}{2\Pi}\int_{-\infty}^{\infty} e^{-i\beta(z-\frac{n^2}{y})}(1-\frac{2i\beta\sigma^2}{a^2})^{-\frac{n-1}{2}}d\beta$$

$$= \frac{nae^{\frac{na\mu}{\sigma^2}}}{\sqrt{2\Pi\sigma^2}}e^{-\frac{n^2a^2}{2\sigma^2y}-\frac{\mu^2y}{2\sigma^2}}y^{-1\frac{1}{2}}$$

$$\frac{(\frac{a^2}{2\sigma^2})^{\frac{n-1}{2}}}{\Gamma(\frac{n-1}{2})}e^{\frac{a^2}{2\sigma^2}(z-\frac{n^2}{y})}(z+\frac{n^2}{y})^{\frac{n-3}{2}}$$

Again putting $U=\bar{X}=\frac{v}{n}$, $V=\tilde{X}=\frac{n}{z}$, we find

$$f_{U,V}(u,v) = \frac{nae^{\frac{na\mu}{\sigma^2}}}{\sqrt{2\Pi\sigma^2}}e^{-\frac{na^2}{2\sigma^2u}-\frac{n\mu^2u}{2\sigma^2}}n^{-1\frac{1}{2}}u^{-1\frac{1}{2}}$$

$$\frac{(\frac{a}{2\sigma^2})^{\frac{n-1}{2}}}{\Gamma(\frac{n-1}{2})}e^{-\frac{na^2}{2\sigma^2}(\frac{1}{v}-\frac{1}{u})}(\frac{1}{v}-\frac{1}{u})^{\frac{n-3}{2}}n^{\frac{n-3}{2}}n^2v^{-2}$$

$$= \frac{\sqrt{n}ae^{\frac{na\mu}{\sigma^2}}}{\sqrt{2\Pi\sigma^2}}e^{-\frac{na^2}{2\sigma^2u}-\frac{n\mu^2u}{2\sigma^2}}u^{-1\frac{1}{2}}$$

$$\frac{(\frac{na^2}{2\sigma^2})^{\frac{n-1}{2}}}{\Gamma(\frac{n-1}{2})}e^{-\frac{na^2}{2\sigma^2}(\frac{1}{v}-\frac{1}{u})}(\frac{1}{v}-\frac{1}{u})^{\frac{n-3}{2}} \qquad \ldots\ldots (4.4)$$

Letting $S=\frac{1}{u}$, $W=\frac{1}{v}$, $u=\frac{1}{s}$ and $v=\frac{1}{w}$

$$\left|\frac{\partial(u,v)}{\partial(s,w)}\right| = \frac{1}{s^2}\cdot\frac{1}{w^2} \quad ;$$

from (4.4)

$$f_{S,W}(s,w) = \frac{\sqrt{n}a e^{\frac{na\mu}{\sigma^2}}}{\sqrt{2\Pi\sigma^2}} e^{-\frac{na^2 s}{2\sigma^2}-\frac{n\mu^2}{2\sigma^2 s}} s^{1\frac{1}{2}}$$

$$\cdot\frac{(\frac{a}{2\sigma^2})^{\frac{n-1}{2}}}{\Gamma(\frac{n-1}{2})} e^{-\frac{na^2}{2\sigma^2}(w-s)} (w-s)^{\frac{n-3}{2}} n^{\frac{n-3}{2}} n^2 w^2 \frac{1}{s^2}\,\frac{1}{w^2}$$

$$= \frac{\sqrt{n}a e^{\frac{na\mu}{\sigma^2}}}{\sqrt{2\Pi\sigma^2}} e^{-\frac{na^2 s}{2\sigma^2}-\frac{n\mu^2}{2\sigma^2 s}} s^{-\frac{1}{2}}$$

$$\cdot\frac{(\frac{na^2}{2\sigma^2})^{\frac{n-1}{2}}}{\Gamma(\frac{n-1}{2})} e^{-\frac{na^2}{2\sigma^2}(w-s)} (w-s)^{\frac{n-3}{2}} \qquad \ldots\ldots (4.5)$$

From (4.6)

$$f_{\frac{\Sigma x}{n}}(u) = \frac{\sqrt{n}a e^{\frac{na\mu}{\sigma^2}}}{\sqrt{2\Pi\sigma^2}} e^{-\frac{na^2}{2\sigma^2 u}-\frac{n\mu^2 u}{2\sigma^2}} u^{-1\frac{1}{2}} \;.$$

From $\phi(\alpha,\beta)$ we have that

$$\phi_{\Sigma\mathbf{x}}(\alpha) = \phi_{\Sigma\mathbf{x},\Sigma\frac{1}{\mathbf{x}}}(\alpha,0) = e^{\frac{na\mu}{\sigma^2}-\frac{na}{\sigma^2}\sqrt{\mu^2-2i\alpha\sigma^2}}$$

$$\text{so } f_{\Sigma\mathbf{x}}(y) = \frac{e^{\frac{na\mu}{\sigma^2}}}{2\Pi}\int_{-\infty}^{\infty} e^{-\frac{na}{\sigma^2}\sqrt{\mu^2-2i\alpha\sigma^2}\,-i\alpha y}\,d\alpha$$

$$= e^{\frac{na\mu}{\sigma}}\frac{na}{\sqrt{2\Pi\sigma^2}}e^{-\frac{n^2a^2}{2\sigma^2u}-\frac{\mu^2u}{2\sigma^2}}u^{-1\frac{1}{2}} \qquad \text{...... (4.6)}$$

and on letting Q=W-S, in (4.5), $\frac{dw}{dq}=1$ where $W=\frac{1}{v}$ and $S=\frac{1}{u}$

$$\text{so } f_{S,Q}(s,q) = \frac{nae^{\frac{na\mu}{\sigma^2}}}{\sqrt{2\Pi\sigma^2}}e^{-\frac{na^2s}{2\sigma^2}-\frac{n\mu^2}{2\sigma^2s}}s^{-\frac{1}{2}}$$

$$\cdot\frac{\left(\frac{na^2}{2\sigma^2}\right)^{\frac{n-1}{2}}}{\Gamma\left(\frac{n-1}{2}\right)}e^{-\frac{na^2}{2\sigma^2}q}\,q^{\frac{n-3}{2}} \qquad \text{...... (4.7)}$$

$$\text{thus } f_Q(q) = \frac{\left(\frac{na^2}{2\sigma^2}\right)^{\frac{n-1}{2}}}{\Gamma\left(\frac{n-1}{2}\right)}e^{-\frac{na^2q}{2\sigma^2}}\,q^{\frac{n-3}{2}}$$

$$\cdot\frac{\sqrt{n}ae^{\frac{na\mu}{\sigma^2}}}{\sqrt{2\Pi\sigma^2}}\int_{0}^{\infty} e^{-\frac{na^2s}{2\sigma^2}-\frac{n\mu^2}{2\sigma^2s}}s^{-\frac{1}{2}}ds$$

$$= \frac{(\frac{na^2}{2\sigma^2})^{\frac{n-1}{2}}}{\Gamma(\frac{n-1}{2})} e^{-\frac{na^2}{2\sigma^2}q} \; q^{\frac{n-3}{2}} \; .1$$

From (4.4) we therefore have that U and $Q=\frac{1}{V}-\frac{1}{U}$ are independent i.e. $\bar{X}$ and $\frac{1}{\tilde{X}}-\frac{1}{\bar{X}}$ are independent, Q being a gamma variate. When sampling from

$$f_X(x) = \frac{\mu e^{\frac{a\mu}{\sigma^2}}}{\sqrt{2\Pi\sigma^2}} e^{-\frac{a^2}{2\sigma^2 x}-\frac{\mu^2 x}{2\sigma^2}} x^{-\frac{1}{2}},$$

we found that $\tilde{X}$ and $\bar{X}-\tilde{X}$ were independent and when sampling from

$$f_X(x) = \frac{a e^{\frac{a\mu}{\sigma^2}}}{\sqrt{2\Pi\sigma^2}} e^{-\frac{a^2}{2\sigma^2 x}-\frac{\mu^2 x}{2\sigma^2}} x^{-1\frac{1}{2}} ,$$

that $\bar{X}$ and $\frac{1}{\tilde{X}}-\frac{1}{\bar{X}}$ were independent. It would be interesting to investigate similar independences for similar sampled distributions viz. distributions having x^{-m} instead of $x^{-1\frac{1}{2}}$ and $x^{-\frac{1}{2}}$. In the case with $x^{-\frac{1}{2}}$, $\tilde{X}$ is a measure of location and $\bar{X}-\tilde{X}$ a measure of dispersion. The p.d.f. of the ratio $\frac{\tilde{X}}{\bar{X}-\tilde{X}}$ is easy to find:

Let $Z=\frac{V}{W}$ in (4.3), then $v=wz$ and $\frac{dv}{dz}=w$,

$$\text{so } f_{W,Z}(w,z) = \frac{\sqrt{n}\; e^{\frac{na}{\sigma^2}}}{\sqrt{2\Pi\sigma^2}} e^{-\frac{na^2}{2\sigma^2 zw}-\frac{w}{2\sigma^2}(nz-na)} \; w^{\frac{n-2}{2}} z^{-\frac{1}{2}}$$

$$\text{so } f_Z(z) = \frac{\sqrt{n}\, e^{\frac{na}{\sigma^2}}}{\sqrt{2\Pi\sigma^2}} z^{-\frac{1}{2}} \int_0^{\infty} e^{-\frac{na^2}{2\sigma^2 zw} - \frac{w}{2\sigma^2}(nz-na)} w^{\frac{n}{2}-1} dw$$

$$= \frac{\sqrt{n}\, e^{\frac{na}{\sigma^2}}}{\sqrt{2\Pi\sigma^2}} 2a^{\frac{n}{2}} \left(\frac{z-a}{z}\right)^{\frac{n}{2}} K_{\frac{n}{2}}\left(\frac{na\sqrt{z-a}}{\sigma^2\sqrt{z}}\right) .$$

However we would like to device a test similar to the t-test viz. one using $\frac{\tilde{X}-k}{\bar{\bar{X}}-\tilde{X}}$ as test statistic, the idea being that null-hypothesis $k=k_o$ would be rejected if $\frac{\tilde{x}-k_o}{\bar{\bar{x}}-\tilde{x}}$ were found to be greater than a certain number. However it does not seem as if the p.d.f. of $\frac{\tilde{X}-k}{\bar{\bar{X}}-\tilde{X}}$ could be found without much difficulty. Also, since the population arithmetic and harmonic means are not much different in value, one may perhaps obtain an inaccurate value for $\bar{x}-\tilde{x}$, perhaps not (since $\bar{x}-\tilde{x}$ is, like the standard deviation, based on all observations). For the p.d.f. with $x^{-1\frac{1}{2}}$, $\bar{X}$ is a measure of location and $\frac{1}{\tilde{X}} - \frac{1}{\bar{\bar{X}}}$ a measure of dispersion. The matter of hypotheses testing received attention in chapter 3. In a sense the distributions with $x^{-\frac{1}{2}}$ are inverses of one another: If these distributions have the same parameters a and μ, then the reciprocal of the variable of each has the 'inverse' distribution with the parameters interchanged:

$$\text{In } f_X(x) = \frac{ae^{\frac{a\mu}{\sigma^2}}}{\sqrt{2\Pi\sigma^2}} e^{-\frac{a^2}{2\sigma^2 x} - \frac{\mu^2 x}{2\sigma^2}} x^{-1\frac{1}{2}}, \text{ put } X=\frac{1}{Y} \text{ , then}$$

$$f_Y(y) = \frac{ae^{\frac{\mu a}{\sigma^2}}}{\sqrt{2\Pi\sigma^2}} e^{-\frac{\mu^2}{2\sigma^2 y} - \frac{a^2 y}{2\sigma^2}} y^{-\frac{1}{2}}$$

4.3 The kappa distribution

Oliver [12], calls the distribution whose p.d.f. is of the general form

$$f(x) = \left(2\left[\frac{a}{b}\right]^{\frac{1}{2}m} K_m(2a^{\frac{1}{2}}b^{\frac{1}{2}})\right)^{-1} e^{-a/x-bx} x^{m-1},$$

$a,b \geq 0$ and $x \geq 0$, the kappa distribution because of the appearance of K, the modified Bessel function of the second kind, the Greek capital K (cf. Γ of the gamma and B of the beta distribution). It is a generalization of the inverse normal and its associated distribution with $x^{-\frac{1}{2}}$ (remembering that

the inverse normal distribution has only two parameters obtainable by maximum likelihood i.e. the above a is equal to $\frac{a^2}{\sigma^2}$ and b to the $\frac{\mu^2}{\sigma^2}$ of the inverse normal distribution). He finds approximate estimators of the parameters a and b. By making use of continued fractions, he shows that for large m the moments of the kappa distribution can be replaced by those of the gamma distribution, that the gamma distribution is a good approximation of the kappa distribution for large m. He also shows that if $A=\Sigma X^2$ and $T=\Sigma X^2 \Sigma Y^2-\Sigma^2 XY$, and a random sample $\{(X_1,Y_1),\ldots,(X_n,Y_n)\}$ is drawn from a bivariate normal $(0,0,1,p,1)$ distribution, then

$$f(a|t) = \frac{1}{\sqrt{2\Pi}} e^{-([\frac{t}{p}]-a)^2/2a}$$

(P the population covariance determinant)

$$= \frac{e^{\sqrt{\frac{t}{p}}}}{\sqrt{2\Pi}} e^{-\frac{t}{2Pa}-\frac{a}{2}} a^{-\frac{1}{2}}$$

which is a special case of the associate of the inverse normal. This statement could probably be adapted to read that the distribution of a sample variance from any bivariate normal distribution, given the covariance determinant, has the associate of the inverse normal as its distribution. There seems to be no reason why Oliver's result could not be extended to sampling from the bivariate normal $(\mu_x, \mu_y, \sigma_x^2, \rho, \sigma_y^2)$ distribution, and going from ΣX^2 and $\Sigma X^2 \Sigma Y^2 - \Sigma^2 XY$ (or $\Sigma(X-\mu_x)^2$ and $\Sigma(X-\mu_x)^2 \Sigma(Y-\mu_y)^2 - \Sigma^2(X-\mu_x)(Y-\mu_y)$) to the sample variance of X and the covariance determinant should just be a matter of division by n-1 and reducing the number of degrees of freedom by 1. The reciprocal of the conditional sample variance should therefore have an inverse normal distribution, and so the inverse normal distribution is not only that of sample size (cf. Wald's work above), and so the sampling theory of the inverse normal distribution need not be a mere mathematical feat.

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